An Adaptive-Sparse Polynomial Dimensional Decomposition Method for High-Dimensional Stochastic Computing

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Outline

Polynomial Dim. Decomposition (Rahman, 2008)

Input $\mathbf{X} \in \mathbb{R}^N \rightarrow$ **MATH MODEL** $\rightarrow$ Output $y(\mathbf{X}) \in \mathbb{R}$

$$\mathbf{X} \sim (\Omega, \mathcal{F}, P);\; y \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$$

- **ANOVA Dimensional Decomposition**

$$y(\mathbf{X}) = \sum_{u \subseteq \{1,\cdots,N\}} y_u(\mathbf{X}_u)$$

- **PDD**

$$y(\mathbf{X}) := y_\emptyset + \sum_{\emptyset \neq u \subseteq \{1,\cdots,N\}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|} \\ j_1,\cdots,j_{|u|} \neq 0}} C_{u\mathbf{j}_{|u|}} \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)$$

$\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u) \rightarrow$ product of univariate orthonormal polynomials

$$y_\emptyset := \int_{\mathbb{R}^N} y(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}; \quad C_{u\mathbf{j}_{|u|}} := \int_{\mathbb{R}^N} y(\mathbf{x}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

PDD Approximation

- **S-variate, mth-order PDD Approximation**

$$\tilde{y}_{S,\boldsymbol{m}_S}(\mathbf{X}) = y_\emptyset + \sum_{\substack{\emptyset \neq u \subseteq \{1,\cdots,N\} \\ 1 \leq |u| \leq S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \|\mathbf{j}_{|u|}\|_\infty \leq m_u \\ j_1,\cdots,j_{|u|} \neq 0}} C_{u\mathbf{j}_{|u|}} \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)$$

An Example ($S = 2$)

$$\tilde{y}_{2,\boldsymbol{m}_2}(\mathbf{X}) = y_\emptyset + \sum_{i=1}^{N}\sum_{j=1}^{m_i} C_{ij}\psi_{ij}(X_i) + \sum_{i_1<i_2}^{N} \sum_{j_2=1}^{m_{i_1 i_2}} \sum_{j_1=1}^{m_{i_1 i_2}} C_{i_1 i_2 j_1 j_2} \psi_{i_1 j_1}(X_{i_1}) \psi_{i_2 j_2}(X_{i_2})$$

- How to select truncation parameters: S, m_u?
- Given S, m_u, terms of all interaction order may not be needed

Global Sensitivity Analysis

- **Sensitivity Indices for $\emptyset \neq u \subseteq \{1, \cdots, N\}$ and m_u**

$$S_{u,m_u} := \frac{\sum\limits_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \|\mathbf{j}_{|u|}\|_\infty \leq m_u \\ j_1, \cdots, j_{|u|} \neq 0}} C_{u\mathbf{j}_{|u|}}^2}{\sum\limits_{\substack{\emptyset \neq u \subseteq \{1, \cdots, N\} \\ 1 \leq |u| \leq S}} \sum\limits_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \|\mathbf{j}_{|u|}\|_\infty \leq m_u \\ j_1, \cdots, j_{|u|} \neq 0}} C_{u\mathbf{j}_{|u|}}^2}$$

$$\triangle \bar{S}_{u,m_u} = \frac{S_{u,m_u} - S_{u,m_u-1}}{S_{u,m_u-1}}$$

- **Adaptive PDD**

$$\tilde{y}(\mathbf{X}) = y_\emptyset + \sum_{\substack{\emptyset \neq u \subseteq \{1, \cdots, N\} \\ S_{u,m_u} > \epsilon_1}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \|\mathbf{j}_{|u|}\|_\infty \leq m_u \\ j_1, \cdots, j_{|u|} \neq 0 \\ \triangle \bar{S}_{u,m_u} > \epsilon_2}} C_{u\mathbf{j}_{|u|}} \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)$$

Coefficient Calculation

- **Dim.-Reduction Integration (Xu and Rahman, 2004)**

$$\hat{y}_R(\boldsymbol{x}) = \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \sum_{\substack{u\subseteq\{1,\cdots,N\}\\ |u|=R-k}} y(\mathbf{x}_u, \mathbf{c}_{-u})$$

$$C_{u\mathbf{j}_{|u|}} \cong \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \sum_{\substack{u\subseteq\{1,\cdots,N\}\\ |u|=R-k}} \times \int_{\mathbb{R}^{|u|}} y(\mathbf{x}_u, \mathbf{c}_{-u}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u) f_{\mathbf{X}_u}(\mathbf{x}_u)\, d\mathbf{x}_u$$

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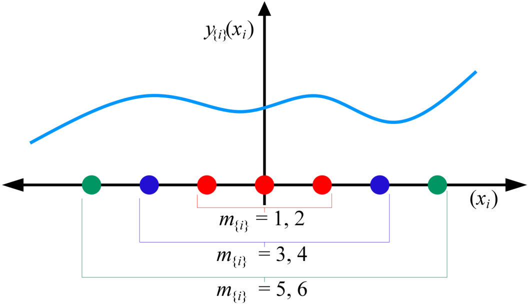
- Highly efficient when $R = S \ll N$ (also convergent)
- Polynomial complexity; 1D or 2D integration for univariate or bivariate approximation

Coefficient Calculation

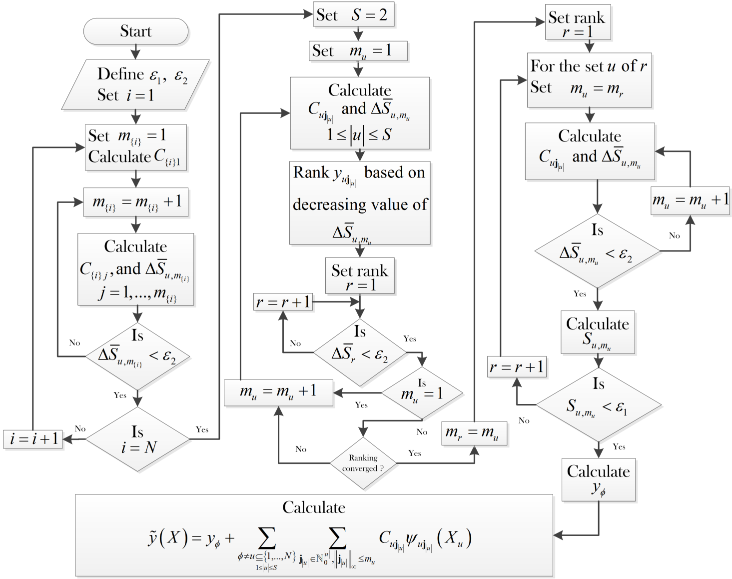
- **Nested Quadrature Rule**

$$y(\mathbf{x}_u, \mathbf{c}_{-u}) \cong \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \|\mathbf{j}_{|u|}\|_\infty \leq n_u \\ j_1, \cdots, j_{|u|} \neq 0}} \phi_{\mathbf{j}_{|u|}}(\mathbf{x}_u) y(\mathbf{x}_u^{(j_{|u|})}, \mathbf{c}_{-u})$$



Function evaluations can be reused as integration points increase

Flowchart



Start
Define ε_1, ε_2 Set $i=1$
Set $m_{\{i\}}=1$ Calculate $C_{\{i\}1}$
$m_{\{i\}}=m_{\{i\}}+1$
Calculate $C_{\{i\}j}$, and $\Delta\bar{S}_{u,m_{\{i\}}}$ $j=1,\ldots,m_{\{i\}}$
Is $\Delta\bar{S}_{u,m_{\{i\}}}<\varepsilon_2$
No
Yes
Is $i=N$
No
Yes
$i=i+1$
Set $S=2$
Set $m_u=1$
Calculate $C_{u\mathbf{j}_{|u|}}$ and $\Delta\bar{S}_{u,m_u}$ $1\le|u|\le S$
Rank $y_{u\mathbf{j}_{|u|}}$ based on decreasing value of $\Delta\bar{S}_{u,m_u}$
Set rank $r=1$
$r=r+1$
Is $\Delta\bar{S}_r<\varepsilon_2$
No
Yes
Is $m_u=1$
Yes
No
$m_u=m_u+1$
Ranking converged ?
No
Yes
$m_r=m_u$
Set rank $r=1$
For the set u of r Set $m_u=m_r$
Calculate $C_{u\mathbf{j}_{|u|}}$ and $\Delta\bar{S}_{u,m_u}$
$m_u=m_u+1$
Is $\Delta\bar{S}_{u,m_u}<\varepsilon_2$
No
Yes
Calculate S_{u,m_u}
$r=r+1$
Is $S_{u,m_u}<\varepsilon_1$
No
Yes
Calculate y_ϕ
Calculate $\tilde{y}(X)=y_\phi+\sum_{\substack{\phi\ne u\subseteq\{1,\ldots,N\}\\ 1\le|u|\le S}}\sum_{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\|\mathbf{j}_{|u|}\|_\infty\le m_u} C_{u\mathbf{j}_{|u|}}\psi_{u\mathbf{j}_{|u|}}(X_u)$

FGM (SiC-Al) Plate Modal Analysis ($N = 34$)

Random Input

- Particle Vol. Fraction (Beta RF)

$$\begin{aligned} \phi_p(x) &\cong F_p^{-1}\left[\Phi\left(\sum_{i=1}^{28} X_i\sqrt{\lambda_i}\psi_i(x)\right)\right] \\ \mu_p(x) &= 1 - x/L \\ \sigma_p(x) &= (1 - x/L)x/L \\ \Gamma_\alpha(\tau) &= \exp[-|\tau|/(0.125L)] \end{aligned}$$

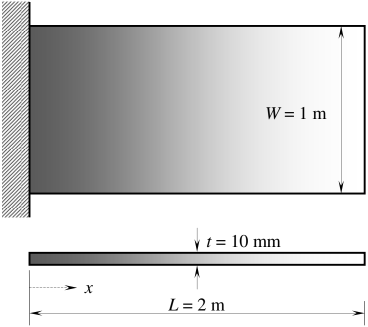
- Constituent Mat. Prop. (RVs)

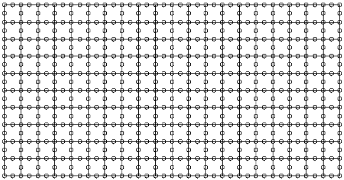
$$\begin{aligned} E(x) &\cong E_p\phi_p(x) + E_m[1 - \phi_p(x)] \\ \nu(x) &\cong \nu_p\phi_p(x) + \nu_m[1 - \phi_p(x)] \\ \rho(x) &\cong \rho_p\phi_p(x) + \rho_m[1 - \phi_p(x)] \end{aligned}$$

$E_p, E_m, \nu_p, \nu_m, \rho_p, \rho_m \rightarrow$ 6 LN variables

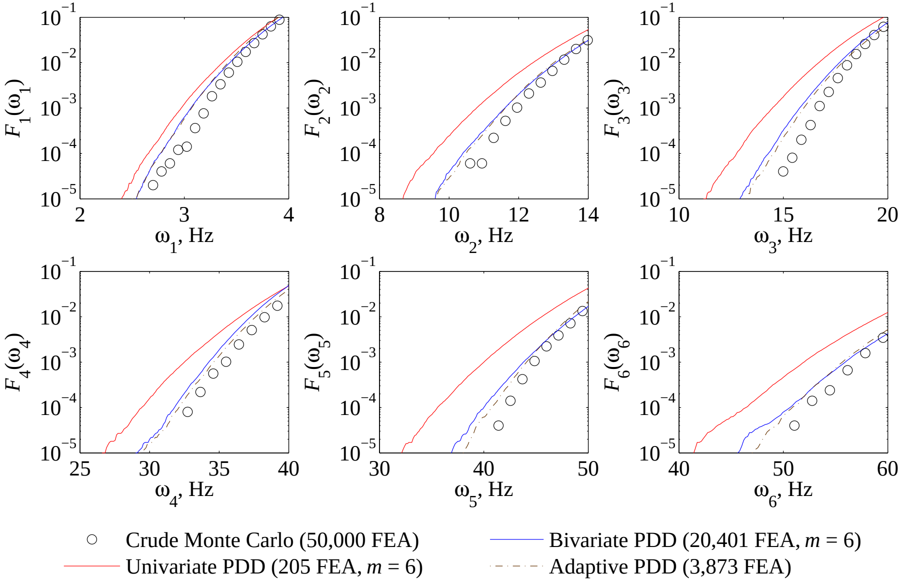
$$\boldsymbol{X} = \{X_1, \cdots, X_{34}\}^T \in \mathbb{R}^{34}$$

Coeff. calc. by dim.-red. integration

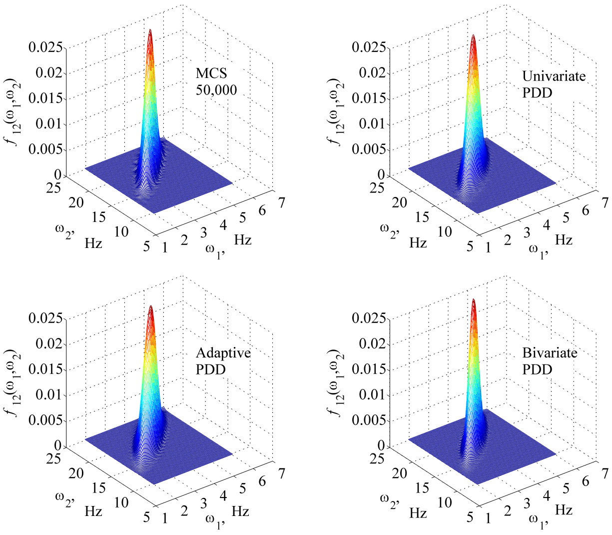


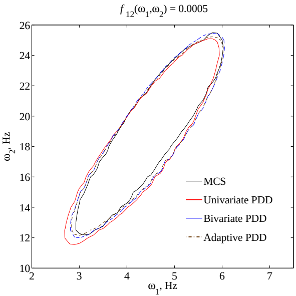


Marginal Distributions of Frequencies



Joint Densities of Frequencies





Method	FEA
Univariate PDD	205
Bivariate PDD	20,401
Adaptive PDD	3,873
MCS	50,000

Conclusions

- An adaptive algorithm for PDD developed
- Global sensitivity indices employed to retain component functions
- Adaptive PDD is significantly more efficient than full bivariate PDD