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Stochastic Computing by a New Polynomial Dimensional Decomposition Method

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OUTLINE

- Introduction
- Polynomial Dimensional Decomposition (PDD) Method
- Examples
- Conclusions & Future Work

INTRODUCTION

- Dimensional Decomposition (Hoeffding, 1948)

Input $\boldsymbol{X} \in \mathbb{R}^N$ ⟶ NONLINEAR SYSTEM ⟶ **Output $y(\boldsymbol{X}) \in \mathbb{R}$**

$$(\Omega, \mathcal{F}) \xrightarrow{\boldsymbol{X}} (\mathbb{R}^N, \mathcal{B}^N) \xrightarrow{y} (\mathbb{R}, \mathcal{B})$$

$$\begin{aligned} y(\boldsymbol{x}) &= y_0 + \sum_{i=1}^{N} y_i(x_i) + \sum_{i_1,i_2=1;\, i_1<i_2}^{N} y_{i_1 i_2}(x_{i_1}, x_{i_2}) + \sum_{i_1,i_2,i_3=1;\, i_1<i_2<i_3}^{N} y_{i_1 i_2 i_3}(x_{i_1}, x_{i_2}, x_{i_3}) \\ &\quad + \cdots + \sum_{i_1,\cdots,i_S=1;\, i_1<\cdots<i_S}^{N} y_{i_1 \cdots i_S}(x_{i_1}, \cdots, x_{i_S}) + \cdots + y_{12\cdots N}(x_1, \cdots, x_N) \end{aligned}$$

(ANOVA or HDMR)

$$\tilde{y}_S(\boldsymbol{x}) = \underbrace{y_0 + \sum_{i=1}^{N} y_i(x_i) + \sum_{i_1,i_2=1;\, i_1<i_2}^{N} y_{i_1 i_2}(x_{i_1}, x_{i_2}) + \cdots + \sum_{i_1,\cdots,i_S=1;\, i_1<\cdots<i_S}^{N} y_{i_1 \cdots i_S}(x_{i_1}, \cdots, x_{i_S})}_{S\text{-variate approximation of } y(\boldsymbol{x})}$$

INTRODUCTION

- Existing Component Functions (Xu/Rahman)

$$y_0 \simeq y(\mathbf{c})$$

ref. point

Lagrange shape function

$$y_i(x_i) \simeq \sum_{j=1}^{n} \phi_j(x_i) y(c_1, \cdots, c_{i-1}, x_i^{(j)}, c_{i+1}, \cdots, c_N)$$

$$y_{i_1,i_2}(x_{i_1}, x_{i_2}) \simeq \sum_{j_2=1}^{n} \sum_{j_1=1}^{n} \phi_{j_1}(x_{i_1}) \phi_{j_2}(x_{i_2}) y(c_1, \cdots, c_{i_1-1}, x_{i_1}^{(j_1)}, c_{i_1-1}, \cdots, c_{i_2-1}, x_{i_2}^{(j_2)}, c_{i_2+1}, \cdots, c_N)$$

$$y_{i_1,\cdots,i_S}(x_{i_1}, \cdots, x_{i_S}) \simeq \sum_{k=0}^{S} (-1)^k \binom{N-S+k-1}{k} \sum_{j_{S-k}=1}^{n} \cdots \sum_{j_1=1}^{n} \phi_{j_1}(X_{i_1}) \cdots \phi_{j_{S-k}}(X_{i_{S-k}}) y(c_1, \cdots, c_{i_1-1}, x_{i_1}^{(j_1)}, c_{i_1+1}, \cdots, c_{i_{S-k}-1}, x_{i_{S-k}}^{(j_{S-k})}, c_{i_{S-k}+1}, \cdots, c_N)$$

Two Weaknesses:

◇ **Arbitrary selection of reference point ($\mathbf{c}$)**

◇ **Arbitrary selection of sample points ($x_i^{(j)}$)**

PDD METHOD

- Orthonormal (ON) Polynomial Basis

$$\boldsymbol{X} = \{X_1, \cdots, X_N\}^T;\ X_i \sim f_i(x_i) \text{ on } (\Omega_i, \mathcal{F}_i, P_i) \text{ \& indep.}$$

$$\mathcal{L}_2(\Omega_i, \mathcal{F}_i, P_i) := \left\{ y_i(x_i) :\ \int_{\mathbb{R}} y_i^2(x_i) f_i(x_i) dx_i < \infty \right\}$$

Hilbert Space

ON Set: $\psi_j(x_i), j = 0, 1, \cdots$

$$\mathbb{E}_{P_i}[\psi_j(X_i)] = \int_{\mathbb{R}} \psi_j(x_i) f_i(x_i) dx_i = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{if } j \neq 0 \end{cases}$$

$$\mathbb{E}_{P_i}[\psi_{j_1}(X_i)\psi_{j_2}(X_i)] = \int_{\mathbb{R}} \psi_{j_1}(x_i)\psi_{j_2}(x_i) f_i(x_i) dx_i = \begin{cases} 1, & \text{if } j_1 = j_2 \\ 0, & \text{if } j_1 \neq j_2 \end{cases}$$

PROPOSED METHOD

- Classical Orthogonal Polynomials

	Hermite	Legendre	Jacobi[a]
Symbol	$H_i(x),\ i=0,\cdots,\infty$	$L_i(x),\ i=0,\cdots,\infty$	$P_i(x;\alpha,\beta),\ i=0,\cdots,\infty$ $\alpha>-1,\ \beta>-1$
Support, $[a,b]$	$[-\infty,+\infty]$	$[-1,+1]$	$[-1,+1]$
Weight function, $w(x)$	$\exp(-x^2/2)$	1	$(1-x)^\alpha(1+x)^\beta$
Probability density function, $f(x)$	$\dfrac{\exp(-x^2/2)}{\sqrt{2\pi}}$ (Gaussian)	$\dfrac{1}{2}$ (Uniform)	$\dfrac{\Gamma(\alpha+\beta+2)(1-x)^\alpha(1+x)^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+1}}$ (Beta)
Differential equation	$H_i''-xH_i'+iH_i=0$	$(1-x^2)L_i''-2xL_i'$ $+i(i+1)L_i=0$	$(1-x^2)P_i''-$ $\beta-\alpha-(\alpha+\beta+2)xP_i'$ $+i(i+\alpha+\beta+1)P_i=0$
Rodrigues′ formula	$H_i=\dfrac{\dfrac{d^i}{dx^i}[\exp(-ix^2/2)]}{(-1)^i\exp(-x^2/2)}$	$L_i=\dfrac{\dfrac{d^i}{dx^i}[(1-x^2)^i]}{(-1)^i2^ii!}$	$P_i=\dfrac{\dfrac{d^i}{dx^i}[(1-x)^{i+\alpha}(1+x)^{i+\beta}]}{(-1)^i2^ii!(1-x)^\alpha(1+x)^\beta}$
Recurrence relation	$H_{i+1}=xH_i-iH_{i-1}$	$(i+1)L_{i+1}=$ $(2i+1)xL_i-iL_{i-1}$	$2(i+1)(i+\alpha+\beta+1)\times$ $(2i+\alpha+\beta)P_{i+1}=$ $[(2i+\alpha+\beta+1)(\alpha^2-\beta^2)+$ $(2i+\alpha+\beta)_3x]P_i-2(i+\alpha)\times$ $(i+\beta)(2i+\alpha+\beta+2)P_{i-1}$
Orthogonality w.r.t. density	$\int_{-\infty}^{+\infty}H_i(x)H_j(x)f(x)dx$ $=i!\delta_{ij}$	$\int_{-\infty}^{+\infty}L_i(x)L_j(x)f(x)dx$ $=\dfrac{1}{2i+1}\delta_{ij}$	$\int_{-\infty}^{+\infty}P_i(x;\alpha,\beta)P_j(x;\alpha,\beta)f(x)dx$ $=\dfrac{2^{\alpha+\beta+1}}{2i+\alpha+\beta+1}\times$ $\dfrac{\Gamma(i+\alpha+1)\Gamma(i+\beta+1)}{i!\Gamma(i+\alpha+\beta+1)}\times$ $\dfrac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+1}}\delta_{ij}$

PROPOSED METHOD

- Fourier-Polynomial Expansions

Coefficient

$$y_i(x_i) = \sum_{j=1}^{\infty} \alpha_{ij}\psi_j(x_i)$$

Coefficient

$$y_{i_1 i_2}(x_{i_1}, x_{i_2}) = \sum_{j_2=1}^{\infty}\sum_{j_1=1}^{\infty} \beta_{i_1 i_2 j_1 j_2}\psi_{i_1 j_1}(x_{i_1})\psi_{i_2 j_2}(x_{i_2})$$

Tensor product

$$\vdots$$

Coefficient

$$y_{i_1 \cdots i_S}(x_{i_1}, \cdots, x_{i_S}) = \sum_{j_S=1}^{\infty} \cdots \sum_{j_1=1}^{\infty} C_{i_1 \cdots i_S j_1 \cdots j_S} \prod_{k=1}^{S} \psi_{j_k}(x_{i_k})$$

PROPOSED METHOD

- Polynomial Dimensional Decomposition

$$
\begin{aligned}
\tilde{y}_S(\boldsymbol{X}) \cong\ & y_0 + \sum_{i=1}^{N}\sum_{j=1}^{m} \alpha_{ij}\psi_{ij}(X_i) + \sum_{i_1,i_2=1;i_1<i_2}^{N} \sum_{j_2=1}^{m}\sum_{j_1=1}^{m} \beta_{i_1 i_2 j_1 j_2}\psi_{i_1 j_1}(X_{i_1})\psi_{i_2 j_2}(X_{i_2}) \\
& + \cdots + \sum_{i_1,\cdots,i_S=1;i_1<\cdots<i_S}^{N} \sum_{j_S=1}^{m}\cdots\sum_{j_1=1}^{m} C_{i_1\cdots i_S j_1\cdots j_S}\prod_{k=1}^{S}\psi_{i_k j_k}(X_{i_k}),
\end{aligned}
$$

◇ **S-variate polynomial decomposition of $y(\boldsymbol{X})$**

◇ **Surrogate of the exact map $y : \mathbb{R}^N \to \mathbb{R}$**

◇ **Converges (m.s.) to $y(\boldsymbol{X})$ as $m \to \infty$, $S \to N$**

PROPOSED METHOD

- Formulation of Coefficients

$$y_0 := \int_{\mathbb{R}^N} y(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \longrightarrow \textbf{Constant}$$

$$\alpha_{ij} := \int_{\mathbb{R}^N} y(\boldsymbol{x}) \psi_{ij}(x_i) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \longrightarrow \textbf{Univariate}$$

$$\beta_{i_1 i_2 j_1 j_2} := \int_{\mathbb{R}^N} y(\boldsymbol{x}) \psi_{i_1 j_1}(x_{i_1}) \psi_{i_2 j_2}(x_{i_2}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \longrightarrow \textbf{Bivariate}$$

$$\vdots$$

$$C_{i_1 \cdots i_S j_1 \cdots j_S} := \int_{\mathbb{R}^N} y(\boldsymbol{x}) \prod_{k=1}^{S} \psi_{i_k j_k}(x_{i_k}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x} \longrightarrow \textbf{\textit{S}-variate}$$

PROPOSED METHOD

- Dimension-Reduction Integration

$$\hat{y}_R(\boldsymbol{x}) := \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \times$$
$$\sum_{k_1,\cdots,k_{R-k}=1;k_1<\cdots<k_{R-k}}^{N} y(c_1,\cdots,c_{k_1-1},x_{k_1},c_{k_1-1},\cdots,c_{k_{R-k}-1},x_{k_{R-k}},c_{k_{R-k}+1},\cdots,c_N)$$

Xu & Rahman, IJNME (2004)

◇ Consists of all terms of the Taylor series of $y(x)$ that have less than or equal to R variables

◇ Residual error contains terms of dimensions $R+1$ and higher

◇ **For $R << N$, an N-dim. integral can be efficiently estimated from an R-dim. integration**

PROPOSED METHOD

- Dimension-Reduction Integration

$$
y_0 \cong \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \sum_{k_1,\cdots,k_{R-k}=1;k_1<\cdots<k_{R-k}}^{N} \times
$$
$$
\int_{\Re^{R-k}} y(c_1,\cdots,c_{k_1-1},x_{k_1},c_{k_1-1},\cdots,c_{k_{R-k}-1},x_{k_{R-k}},c_{k_{R-k}+1},\cdots,c_N) \prod_{s=1}^{R-k} f_{k_s}(x_{k_s})dx_{k_s}
$$

$$
C_{i_1\cdots i_S j_1\cdots j_S} \cong \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \sum_{k_1,\cdots,k_{R-k}=1;k_1<\cdots<k_{R-k}}^{N} \times
$$
$$
\int_{\Re^{R-k}} y(c_1,\cdots,c_{k_1-1},x_{k_1},c_{k_1+1},\cdots,c_{k_{R-k}-1},x_{k_{R-k}},c_{k_{R-k}+1},\cdots,c_N) \times
$$
$$
\prod_{s=1}^{S} \omega_{i_s j_s}(x_{i_s}) \prod_{s=1}^{R-k} f_{k_s}(x_{k_s})dx_{k_s}.
$$

When $R = 1$, 2, or 3, the above equations require one-, at most two-, and at most three-dimensional integrations, respectively

PROPOSED METHOD

- Computational Effort

$$N = \text{No. of random variables}$$
$$n = \text{No. of integration points}$$

Approximation	No. of Function Evaluations	Cost Scaling
Univariate	$nN+1$	Linear
Bivariate	$N(N-1)n^2/2+nN+1$	Quadratic
…	…	…
S-variate	$\sum_{k=0}^{k=S} \binom{N}{S-k} n^{S-k}$	Sth degree

Cost increases polynomically, not exponentially

EXAMPLES

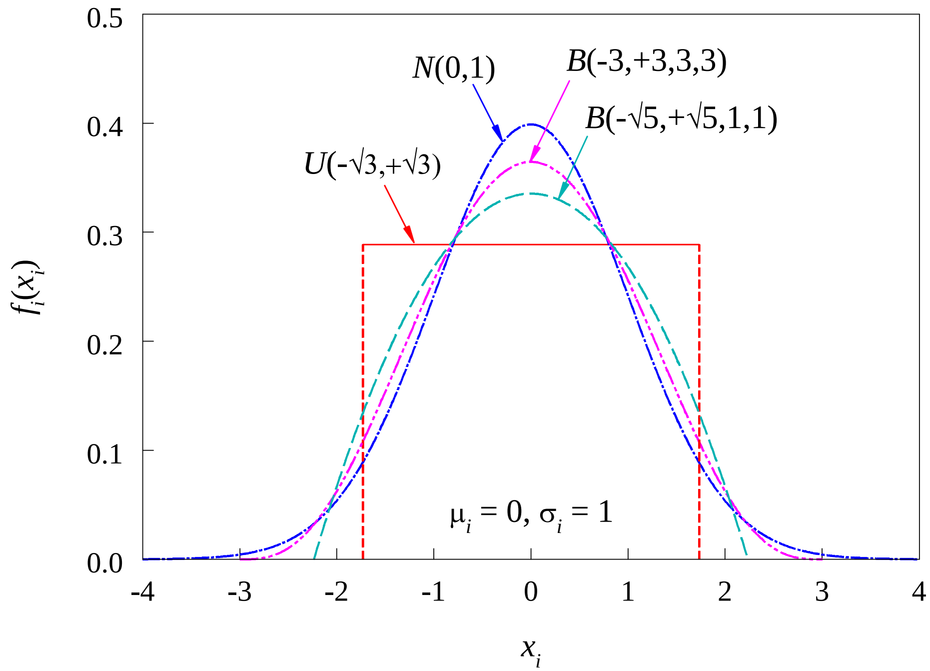
- A Cubic Polynomial

$$y(\boldsymbol{X}) = 50 - (X_1 + X_2)^3 + X_1 - X_2 - X_3 + X_1 X_2 X_3 - X_4$$

$$X_i,\ i = 1 - 4,\ \text{i.i.d, mean } \mu_i = 0,\ \text{st. dev. } \sigma_i = 1$$

Probability Densities of X_i:

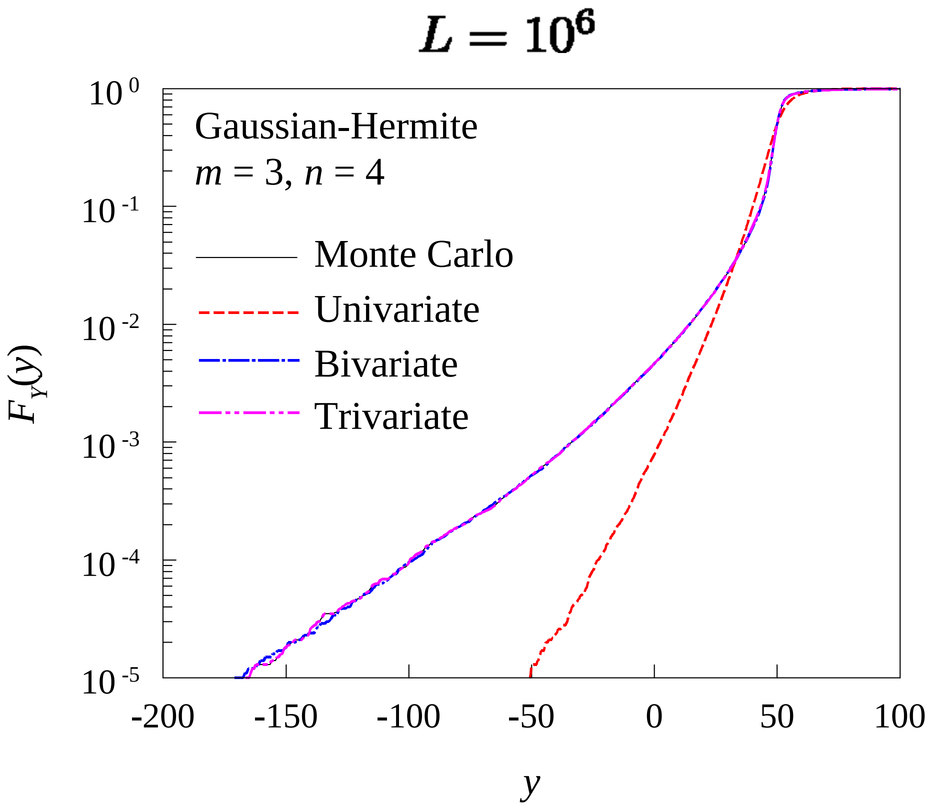
$$f_i(x_i) = \begin{cases} \left(1/\sqrt{2\pi}\right) \exp\left(-x_i^2/2\right), & \text{Gaussian} \\ 1/(2\sqrt{3}), & \text{Uniform} \\ \left[\Gamma(8)/\left\{\Gamma^2(4)6^7\right\}\right] & \\ (x_i+3)^3(3-x_i)^3, & \text{Beta} \\ \left[\Gamma(4)/\left\{\Gamma^2(2)(2\sqrt{5})^3\right\}\right] & \\ (x_i+\sqrt{5})(\sqrt{5}-x_i), & \text{Beta} \end{cases}$$

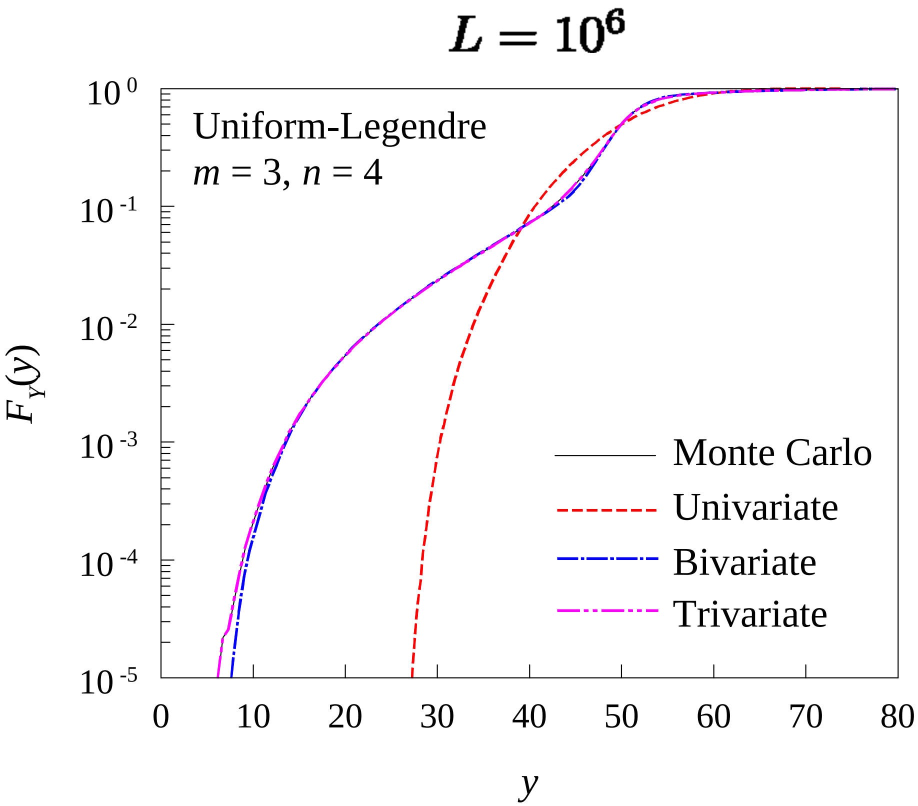


All coefficients calculated by exact numerical integration

EXAMPLES

- Tail Probabilities

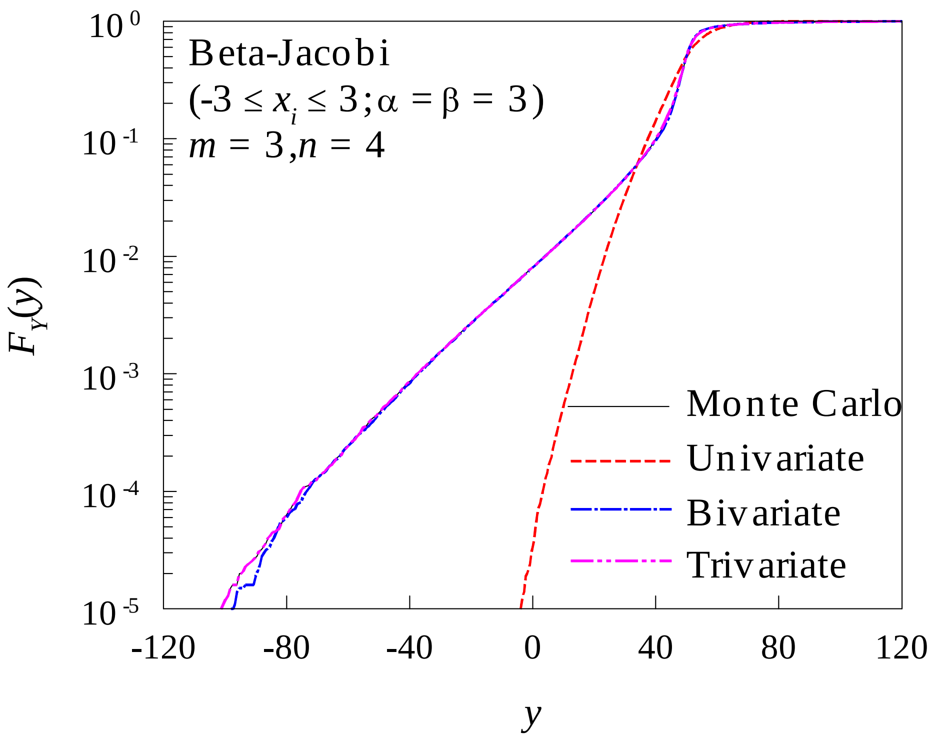




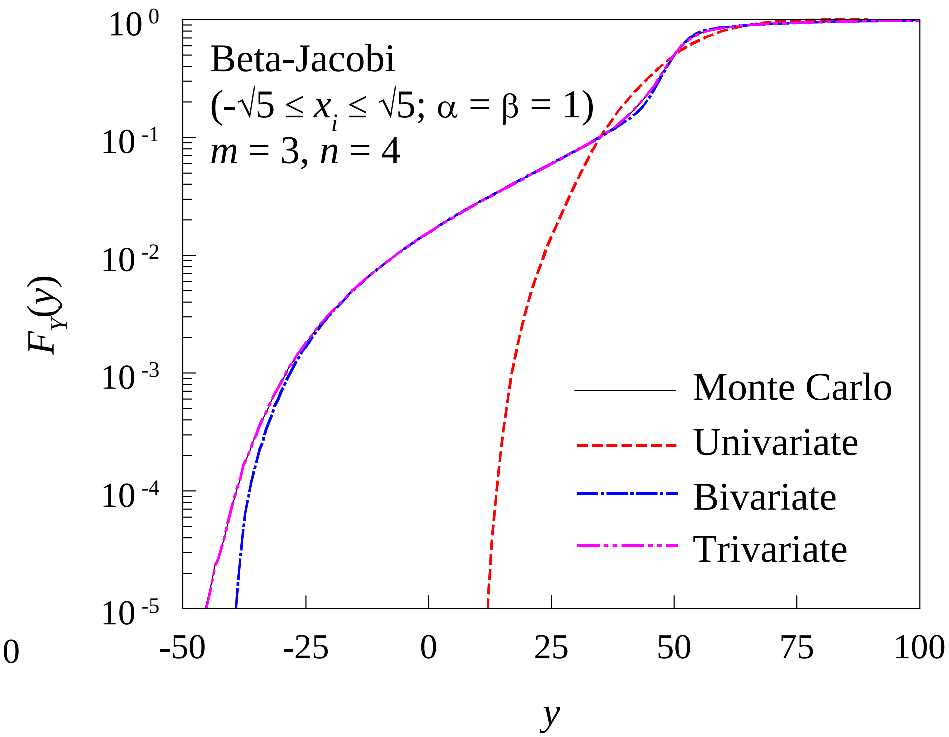
EXAMPLES

- Tail Probabilities

$L = 10^6$

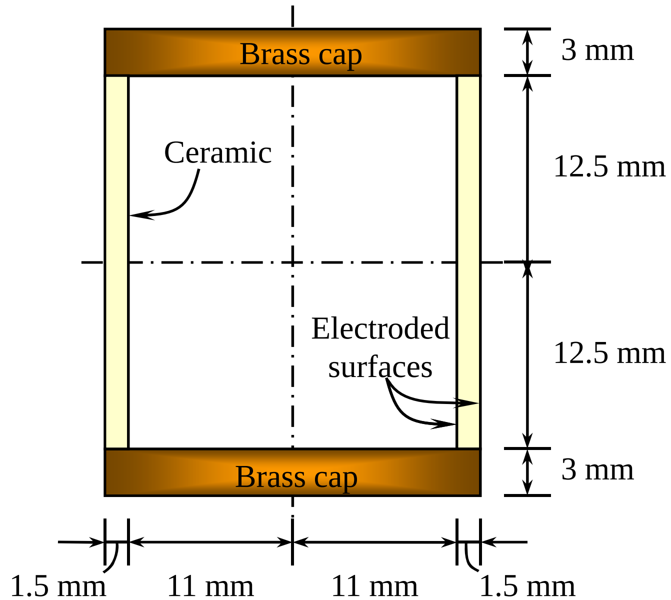


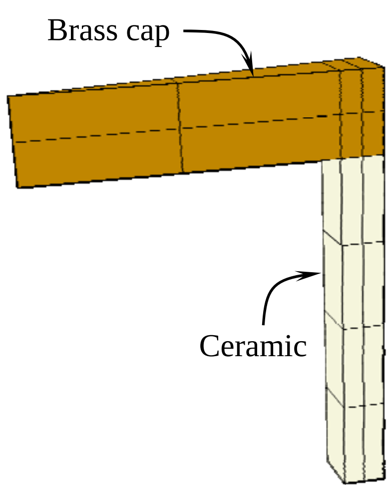
$L = 10^6$



EXAMPLES

- Piezoelectric Transducer





Statistical properties of Input

Random variable	Property[(a)]	Mean	Coefficient of variation
X_1, GPa	D_{1111}	115.4	0.15
X_2, GPa	D_{1122}, D_{1133}	74.28	0.15
X_3, GPa	D_{2222}, D_{3333}	139	0.15
X_4, GPa	D_{2233}	77.84	0.15
X_5, GPa	$D_{1212}, D_{2323}, D_{1313}$	25.64	0.15
X_6, Coulomb/m^2	e_{311}	15.08	0.1
X_7, Coulomb/m^2	e_{322}, e_{133}	-5.207	0.1
X_8, Coulomb/m^2	e_{212}, e_{313}	12.71	0.1
X_9, nF/m	D_{11}	5.872	0.1
X_{10}, nF/m	D_{22}, D_{33}	6.752	0.1
X_{11}, GPa	E_b	104	0.15
X_{12}	ν_b	0.37	0.05
X_{13}, g/m^3	ρ_b	8500	0.15
X_{14}, g/m^3	ρ_c	7500	0.15

(a) D_{ijkl} are elastic moduli of ceramic; e_{ijk} are piezoelectric coupling constants of ceramic; D_{ij} are dielectric constants of ceramic; E_b, ν_b, ρ_b are elastic modulus, Poisson's ratio, and mass density of brass; ρ_c is mass density of ceramic

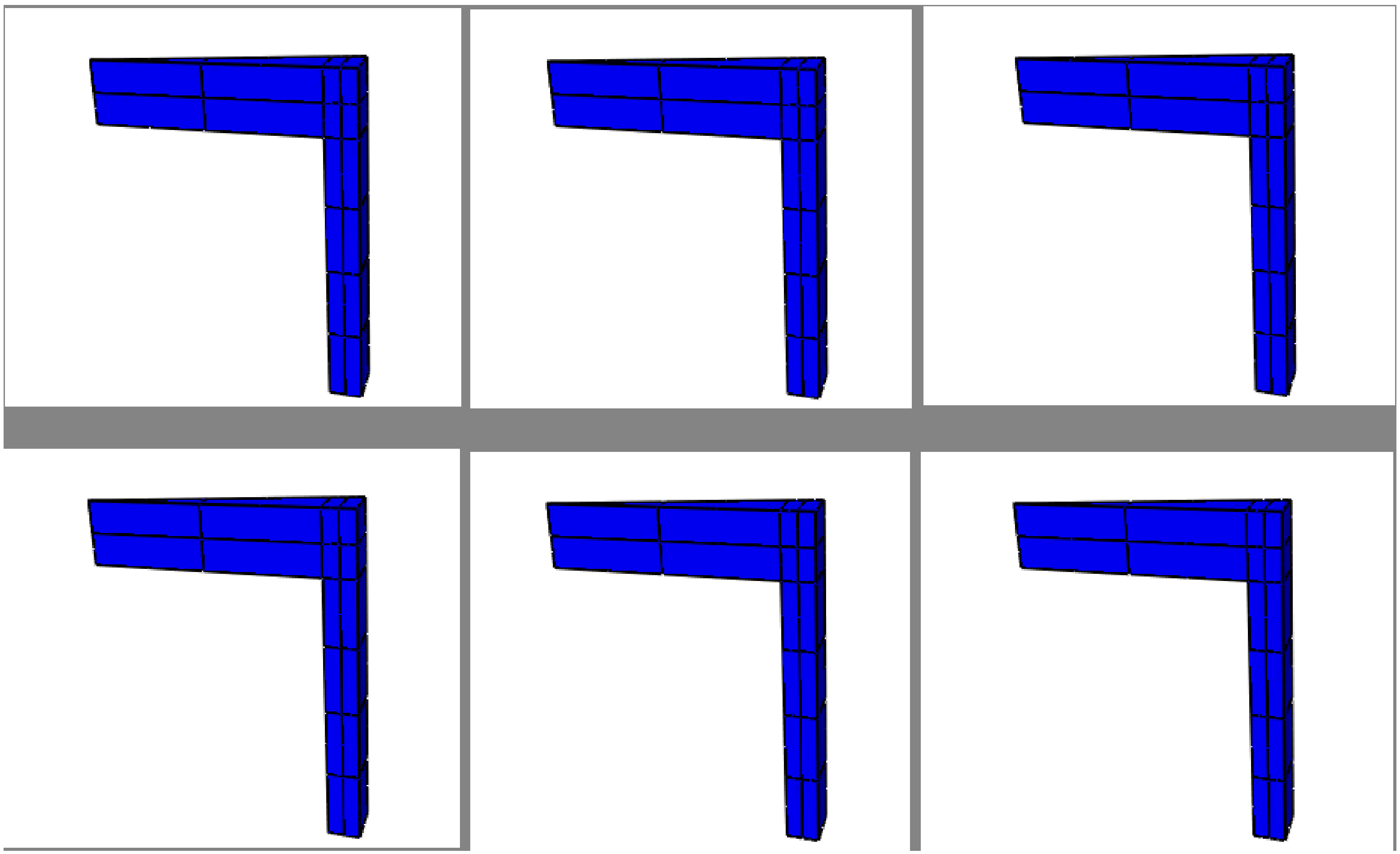
$$m = 3,\ n = 4,\ N = 14,\ L = 10^6$$

All coefficients calculated by S-dim. integ.

EXAMPLES

- A Few Mode Shapes (Mean Input)

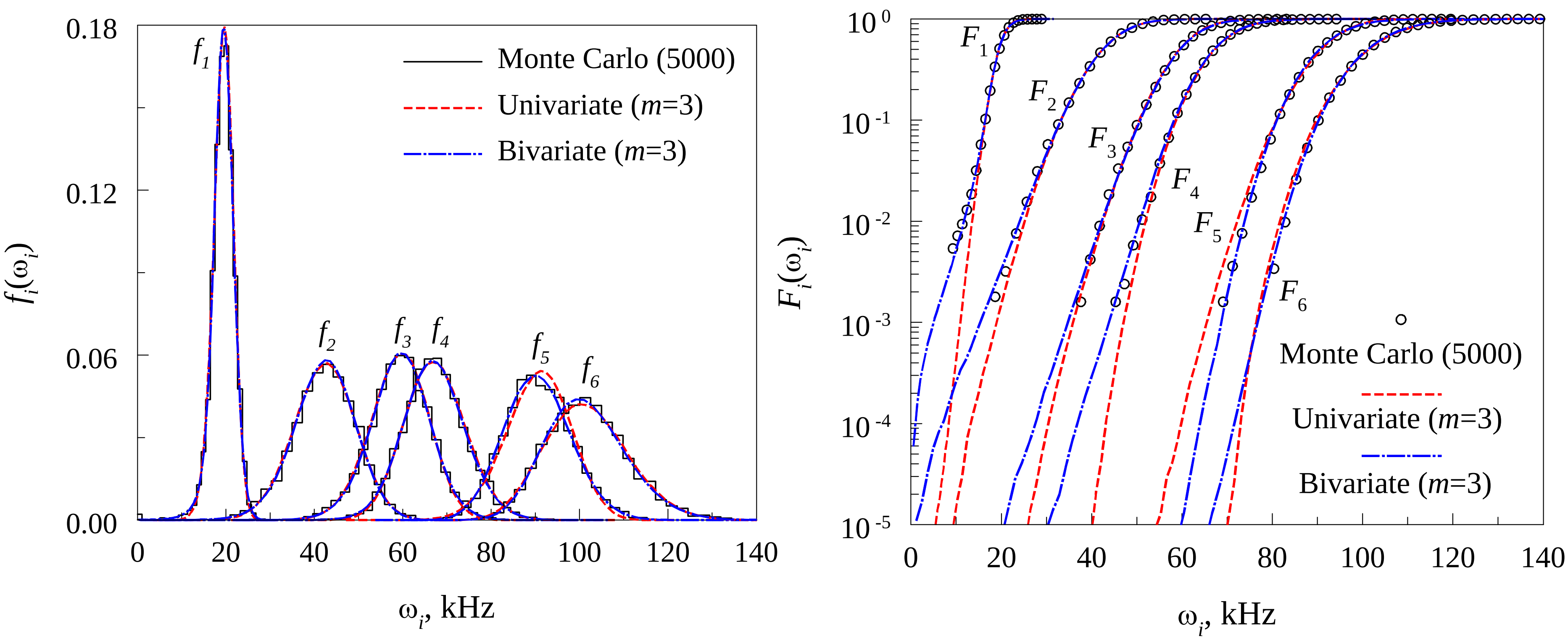
Mode 1 (19.75 kHz) Mode 2 (42.9 kHz) Mode 3 (60.7 kHz)



Mode 4 (66.7 kHz) Mode 5 (92.03 kHz) Mode 6 (100.32 kHz)

EXAMPLES

- Marginal Distributions of Frequencies

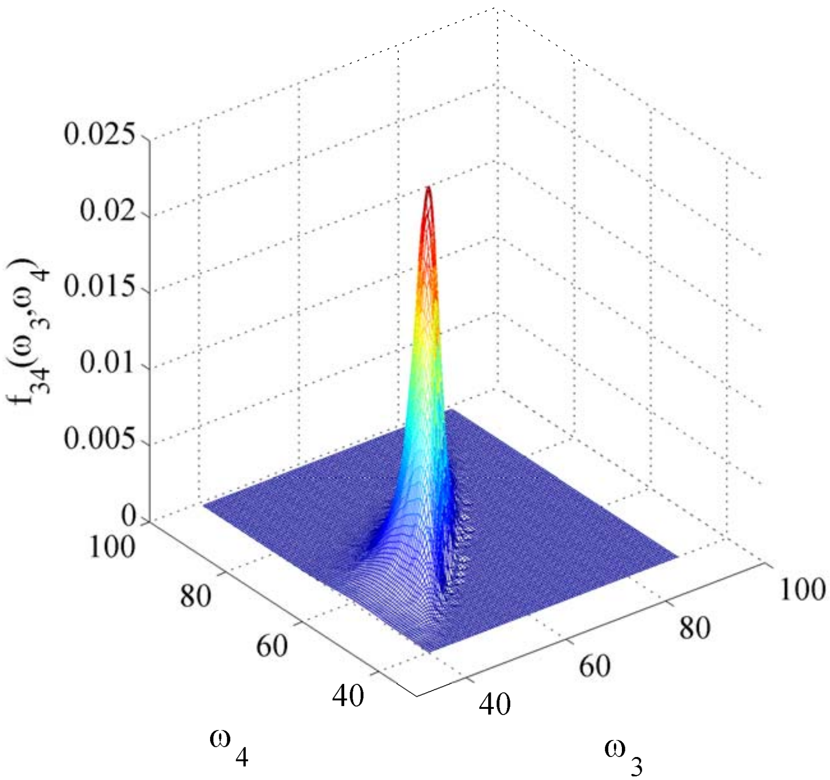


PDD: Univariate (43 FEA); Bivariate (862 FEA)

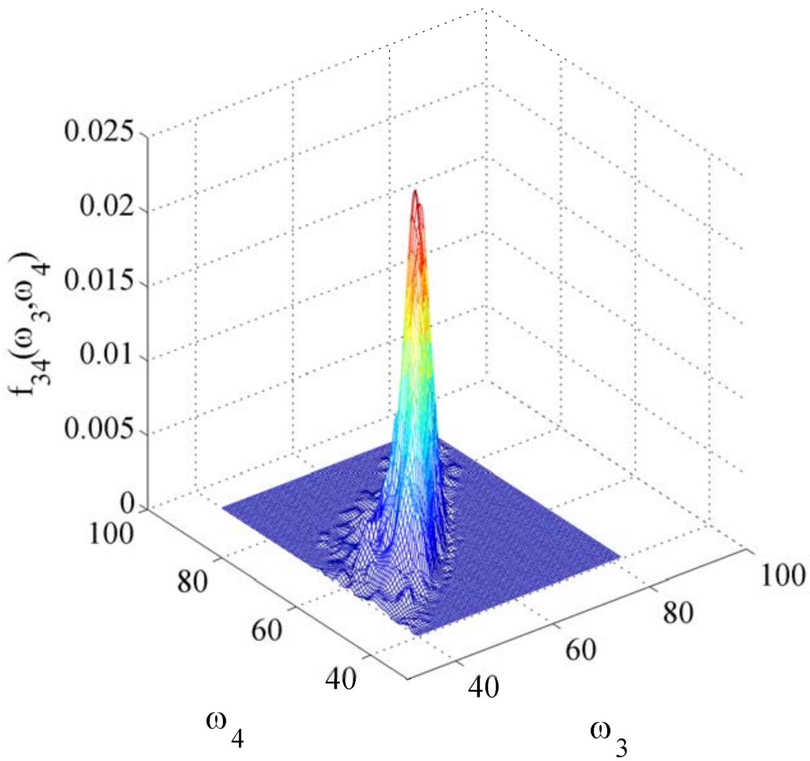
Crude MCS (5000 FEA)

EXAMPLES

- Joint Distributions of Frequencies



Bivariate PDD (862 FEA)



Crude MCS (5000 FEA)

CONCLUSIONS/FUTURE WORK

- A new polynomial dimensional decomposition method was developed
  - Decomposition with increasing dimensions
  - Fourier-polynomial expansions
  - Innovative dimension-reduction integration
- No sample points required; yet, generates convergent approximations
- Accurate with modest computational effort
- Future work: Time-variant problems, arbitrary probability measures, discontinuous & non-smooth responses, *etc.*