RELIABILITY-BASED DESIGN OPTIMIZATION OF LARGE-SCALE COMPLEX SYSTEMS

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INTRODUCTION

- **What is RBDO?**

$$\begin{array}{ll}\min & h_0(\boldsymbol{\theta}) := \mathbb{E}_{\boldsymbol{\theta}}[y_0(\boldsymbol{X})],\\ \text{subject to} & h_k(\boldsymbol{\theta}) := P_{\boldsymbol{\theta}}[\boldsymbol{X} \in \Omega_{F,k}] \le p_k;\ k=1,\cdots,K\end{array}$$

$\boldsymbol{\theta} \in \mathcal{D} \subseteq \mathbb{R}^M \to M$-dim. design vector; $\Omega_{F,k} \to$ failure domain

$\boldsymbol{X} \in \mathbb{R}^N \to N$-dim. input random vector $\sim f_{\boldsymbol{X}}(\boldsymbol{x};\boldsymbol{\theta})$

$\mathbb{E}_{\boldsymbol{\theta}}: f_{\boldsymbol{X}} \to \mathbb{R}$; $P_{\boldsymbol{\theta}}: f_{\boldsymbol{X}} \to [0,1]$; $y_k(\boldsymbol{x}) \to$ performance function

Type	Failure Domain	Application
Component	$\Omega_{F,k} := \{\boldsymbol{x} : y_k(\boldsymbol{x}) < 0\}$	Single failure event
Series System	$\Omega_{F,k} := \{\boldsymbol{x} : \bigcup_q y_k^{(q)}(\boldsymbol{x}) < 0\}$	Union of failure events
Parallel System	$\Omega_{F,k} := \{\boldsymbol{x} : \bigcap_q y_k^{(q)}(\boldsymbol{x}) < 0\}$	Intersection of failure events
General System	$\Omega_{F,k} := \{\boldsymbol{x} : \bigcup_l \bigcap_{q \in C_l} y_k^{(q)}(\boldsymbol{x}) < 0\}$	Certain comb. of fail events

INTRODUCTION

Project Goal

Create new theoretical foundations and numerical algorithms of RBDO methods for large-scale design optimization of complex engineering systems

Project Objectives

- Develop new extended polynomial dimensional decomposition (X-PDD) method for reliability analysis of high-dimensional complex systems (Year 1)
- Integrate X-PDD and score functions for concurrent design sensitivity analysis (Year 2)
- Develop fast and efficient reliability-based and robust design optimization algorithms (Year 3)

(Project started in April 1, 2010)

NEW X-PDD METHOD

- **Dimensional Decomposition (Hoeffding, 1948)**

Input $\boldsymbol{X} \in \mathbb{R}^N \to$ **NONLINEAR SYSTEM** $\to$ Output $y(\boldsymbol{X}) \in \mathbb{R}$

$$(\Omega, \mathcal{F}) \xrightarrow{\boldsymbol{X}} (\mathbb{R}^N, \mathcal{B}^N) \xrightarrow{y} (\mathbb{R}, \mathcal{B})$$

$$y(\boldsymbol{X}) = y_0 + \sum_{i=1}^{N} y_i(X_i) + \sum_{i_1,i_2=1;i_1<i_2}^{N} y_{i_1 i_2}(X_{i_1}, X_{i_2}) + \cdots + \sum_{i_1,\cdots,i_s=1;i_1<\cdots<i_s}^{N} y_{i_1\cdots i_s}(X_{i_1},\cdots,X_{i_s}) + \cdots + y_{12\cdots N}(X_1,\cdots,X_N)$$

$$\underbrace{\tilde{y}_S(\boldsymbol{X}) = y_0 + \sum_{i=1}^{N} y_i(X_i) + \sum_{i_1,i_2=1;i_1<i_2}^{N} y_{i_1 i_2}(X_{i_1}, X_{i_2}) + \cdots + \sum_{i_1,\cdots,i_S=1;i_1<\cdots<i_S}^{N} y_{i_1\cdots i_S}(X_{i_1},\cdots,X_{i_S})}_{S-\text{variate approximation of } y(\boldsymbol{X})}$$

NEW X-PDD METHOD

- **Measure-Consistent ON Polynomial Basis**

$\boldsymbol{X} = \{X_1, \cdots, X_N\}^T$; indep.; $X_i \sim f_i(x_i)$ on (Ω_i, F_i, P_i)

$$\underbrace{\mathcal{L}_2(\Omega_i, \mathcal{F}_i, P_i)}_{\text{Hilbert space}} := \left\{ y_i(x_i) : \int_{\mathbb{R}} y_i^2(x_i) f_i(x_i) dx_i < \infty \right\}$$

$\{\psi_{ij}(x_i);\ j = 0, 1, \cdots\} \to$ a set of ON bases in $\mathcal{L}_2(\Omega_i, \mathcal{F}_i, P_i)$

Two Important Properties of $\psi_{ij}(X_i)$:

$$\mathbb{E}_{P_i}[\psi_{ij}(X_i)] = \int_{\mathbb{R}} \psi_{ij}(x_i) f_i(x_i) dx_i = \begin{cases} 1, & \text{if } j = 0 \\ 0, & \text{if } j \neq 0 \end{cases}$$

$$\mathbb{E}_{P_i}[\psi_{ij_1}(X_i)\psi_{ij_2}(X_i)] = \int_{\mathbb{R}} \psi_{ij_1}(x_i)\psi_{ij_2}(x_i) f_i(x_i) dx_i = \begin{cases} 1, & \text{if } j_1 = j_2 \\ 0, & \text{if } j_1 \neq j_2 \end{cases}$$

NEW X-PDD METHOD

- **Three-Term Recurrence Relation (OG Polynomials)**

$\pi_{i,j+1}(x_i) = (x_i - a_{ij})\pi_{ij}(x_i) - b_{ij}\pi_{i,j-1}(x_i)$; $\psi_{ij}(x_i) := \pi_{ij} / \|\pi_{ij}\|_{dF_i}$

Measure$^{(a)}$	$a_{ij},\ j \ge 0$	$b_{ij},\ j \ge 1$
Gaussian (Hermite)	0	j
Uniform (Legendre)	0	$\frac{1}{(4 - j^{-2})}$
Exponential (Laguerre)	$2j+1$	j^2
Gamma (Gen. Laguerre)	$2j+\alpha+1$	$j(j+\alpha)$
Beta (Jacobi)	$\frac{(\beta^2-\alpha^2)}{(2j+\alpha+\beta)(2j+\alpha+\beta+2)}$	$\frac{4j(j+\alpha)(j+\beta)}{(2j+\alpha+\beta)^2} \times \frac{(j+\alpha+\beta)}{[(2j+\alpha+\beta)^2-1]}$
Lognormal (?)	$\exp\left[\tilde{\mu} + \frac{1}{2}\tilde{\sigma}^2(2j-1)\right] \times \exp(j\tilde{\sigma}^2)\left[\exp(\tilde{\sigma}^2)+1\right] - 1$	$\exp\left[2\tilde{\mu} + \tilde{\sigma}^2(3j-2)\right] \times \left[\exp(j\tilde{\sigma}^2)-1\right]$
Student's t (?)	0	$\frac{j\nu(\nu-j+1)}{(\nu-2j)(\nu-2j+2)}$
Fisher's F (?)	$\frac{(\nu_1\nu_2+2\nu_1+4j\nu_2-8j^2)\times \nu_2/\nu_1}{(\nu_2-4j-2)(\nu_2-4j+2)}$	$\frac{2j(\nu_1+2j-2)(\nu_2-2j+2)}{(\nu_2-4j)(\nu_2-4j+2)^2} \times \frac{(\nu_2/\nu_1)^2(\nu_1+\nu_2-2j)}{(\nu_2-4j+4)}$

(a) For arbitrary measures, Stieltjes procedure was used to calculate a_{ij}, b_{ij}

NEW X-PDD METHOD

- **Measure-Consistent Gauss Quadrature Formula (Golub & Welsch, 1969)**

$$\int_{\mathbb{R}} g_i(x_i) dF_i(x_i) = \int_{\mathbb{R}} g_i(x_i) f_i(x_i) dx_i = \sum_{j=1}^{n} w_i^{(j)} g_i(x_i^{(j)}) + R_{in}(g_i)$$

$$\boldsymbol{J}_{in} := \begin{bmatrix} a_{i0} & \sqrt{b_{i1}} & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{b_{i1}} & a_{i1} & \sqrt{b_{i2}} & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{b_{i2}} & a_{i2} & \sqrt{b_{i3}} & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{b_{i3}} & \ddots & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \sqrt{b_{i,n-2}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{b_{i,n-2}} & a_{i,n-2} & \sqrt{b_{i,n-1}} \\ 0 & 0 & 0 & 0 & 0 & \sqrt{b_{i,n-1}} & a_{i,n-1} \end{bmatrix}$$

- Possesses algebraic degree of exactness $2n-1$
- Obtaining eigensolutions of the Jacobi matrix is trivial

NEW X-PDD METHOD

- **S-variate, m-th order PDD Approximation**

$$\tilde{y}_S(\boldsymbol{X}) = y_0 + \sum_{i=1}^{N}\sum_{j=1}^{m} C_{ij}\psi_{ij}(X_i) + \sum_{i_1,i_2=1;i_1<i_2}^{N} \sum_{j_2=1}^{m}\sum_{j_1=1}^{m} C_{i_1 i_2 j_1 j_2} \prod_{k=1}^{2} \psi_{i_k j_k}(X_{i_k}) + \cdots + \sum_{i_1,\cdots,i_S=1;i_1<\cdots<i_S}^{N} \sum_{j_S=1}^{m} \cdots \sum_{j_1=1}^{m} C_{i_1\cdots i_S j_1\cdots j_S} \prod_{k=1}^{S} \psi_{i_k j_k}(X_{i_k})$$

Fourier-Polynomial Expansions of $y_{i_1\cdots i_s}(X_{i_1},\cdots,X_{i_s})$:

$$y_{i_1\cdots i_s}(X_{i_1},\cdots,X_{i_s}) = \sum_{j_s=1}^{\infty} \cdots \sum_{j_1=1}^{\infty} C_{i_1\cdots i_s j_1\cdots j_s} \prod_{k=1}^{s} \psi_{i_k j_k}(X_{i_k})$$

$$C_{i_1\cdots i_s j_1\cdots j_s} = \int_{\mathbb{R}^N} y(\boldsymbol{x}) \prod_{k=1}^{s} \psi_{i_k j_k}(x_{i_k}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x},\ 1 \le s \le N$$

NEW X-PDD METHOD

- **Dimension-Reduction Integration for Coefficients**

$$\hat{y}_R(\boldsymbol{x}) = \sum_{k=0}^{R} (-1)^k \binom{N-R+k-1}{k} \sum_{i_1,\cdots,i_{R-k}=1;i_1<\cdots<i_{R-k}}^{N} \times y(c_1,\cdots,c_{i_1-1},x_{i_1},c_{i_1+1},\cdots,c_{i_{R-k}-1},x_{i_{R-k}},c_{i_{R-k}+1},\cdots,c_N)$$

$$C_{i_1\cdots i_s j_1\cdots j_s} \cong \sum_{k=0}^{R} (-1)^k \binom{N-R+k-1}{k} \sum_{i_1,\cdots,i_{R-k}=1;i_1<\cdots<i_{R-k}}^{N} \times \int_{\mathbb{R}^{R-k}} y(c_1,\cdots,c_{i_1-1},x_{i_1},c_{i_1+1},\cdots,c_{i_{R-k}-1},x_{i_{R-k}},c_{i_{R-k}+1},\cdots,c_N) \prod_{p=1}^{s}\psi_{i_p j_p}(x_{i_p}) \prod_{p=1}^{R-k} f_{k_p}(x_{k_p}) dx_{k_p}$$

- Highly efficient when $R \ll N$ (also convergent)
- When $R = 1$, 2, or 3, requires one-, at most two-, and at most three-dimensional integrations, respectively

NEW X-PDD METHOD

- **Second-Moment Statistics**

$$\mathbb{E}[\tilde{y}_S(\boldsymbol{X})] = y_0 := \int_{\mathbb{R}^N} y(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}) d\boldsymbol{x}$$

$$\mathbb{E}[\tilde{y}_S(\boldsymbol{X}) - y_0]^2 = \sum_{s=1}^{S}\left(\sum_{i_1,\cdots,i_S=1;i_1<\cdots<i_S}^{N} \sum_{j_s=1}^{m} \cdots \sum_{j_1=1}^{m} C^2_{i_1\cdots i_s j_1\cdots j_s} \right)$$

- **Probability of Failure (Reliability)**

$$P_{\boldsymbol{\theta}}[\boldsymbol{X} \in \Omega_{F,k}] \cong P_{\boldsymbol{\theta}}\left[\boldsymbol{X} \in \tilde{\Omega}_{F,k}\right] = \lim_{L\to\infty} \frac{1}{L} \sum_{l=1}^{L} I_{\tilde{\Omega}_{F,k}}(\boldsymbol{x}^{(l)})$$

$\tilde{\Omega}_{F,k} = S$-variate PDD approximation of $\Omega_{F,k}$

$\boldsymbol{x}^{(l)} = l$th sample of $\boldsymbol{X}$; $I_{\tilde{\Omega}_{F,k}}(\boldsymbol{x}) =$ indicator function

- First two moments provide exact or convergent solutions
- Higher-order moments can be derived, but are less simple

NEW X-PDD METHOD

- **Computational Complexity**

$N =$ no. of random variables; $n =$ no. of integration points

Approximation	No. of Func. Eval.	Cost Scaling
Univariate ($S = 1$)	$nN + 1$	Linear
Bivariate ($S = 2$)	$N(N-1)n^2/2 + nN + 1$	Quadratic
...	...	...
S-variate	$\sum_{k=0}^{k=S} \binom{N}{S-k} n^{S-k}$	Sth degree

- For non-trivial estimates of expansion coefficients, $R \ge S$
- Cost increases polynomically, not exponentially

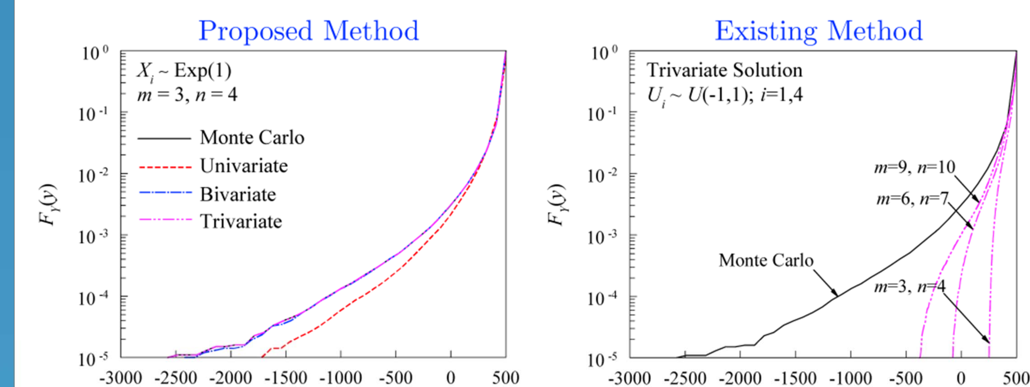
RESULTS

- **Example 1: A Polynomial Function**

$y(\boldsymbol{X}) = 500 - (X_1 + X_2)^3 + X_1 - X_2 - X_3 + X_1X_2X_3 - X_4$

$X_i, i = 1, \cdots, 4$; i.i.d; $\mu_i = \sigma_i = 1$; $f_i(x_i) = \exp(-x_i)$

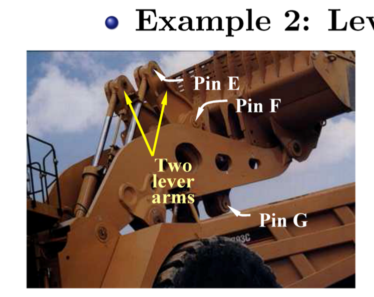
Proposed Method | Existing Method

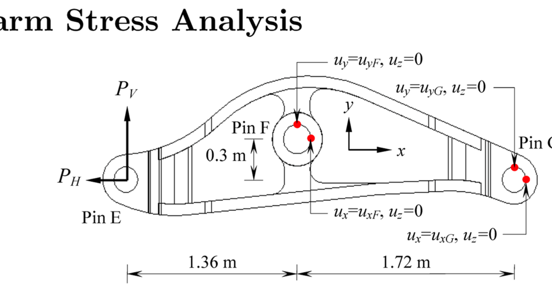


All coefficients calculated by exact num. integration

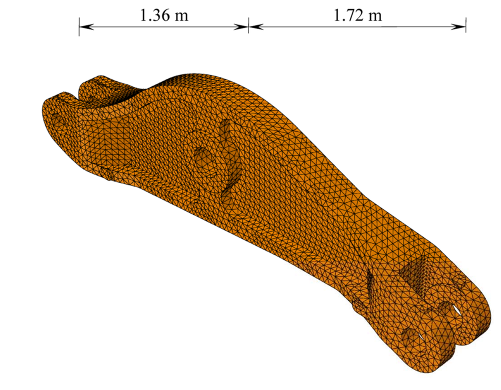
RESULTS

- **Example 2: Leverarm Stress Analysis**





RV	Mean	S.D.	PDF
P_H, kN	507.69	76.15	LN
P_V, kN	1517.32	227.60	LN
E, GPa	203	10.15	LN
ν	0.3	0.015	LN
$u_{E'}$, mm	-5	$5/\sqrt{3}$	U
$u_{F'}$, mm	5	$5/\sqrt{3}$	U
$u_{G'}$, mm	5	$5/\sqrt{3}$	U
$u_{G''}$, mm	-5	$5/\sqrt{3}$	U



All coefficients calculated by dim.-red. integration

RESULTS

- **Second-Moment Properties of Stresses**

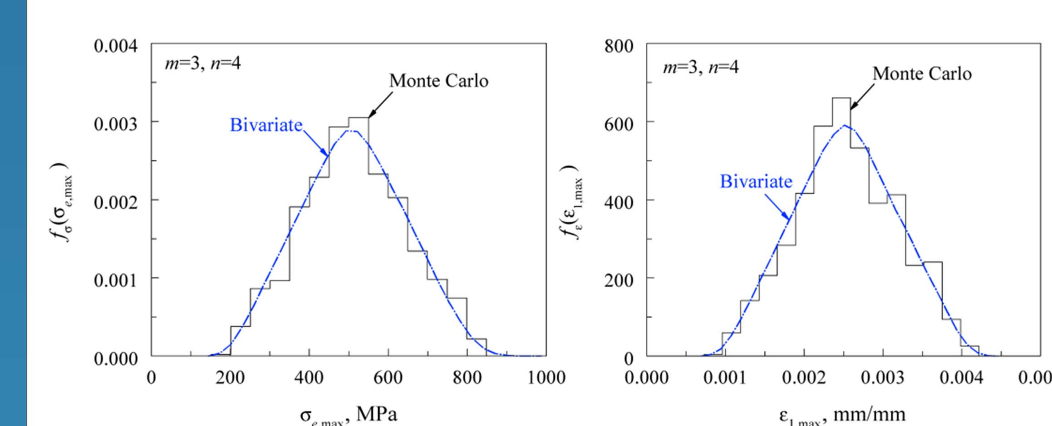
m	Univariate PDD Mean	Univariate PDD S.D.	Univariate PDD FEA	Bivariate PDD Mean	Bivariate PDD S.D.	Bivariate PDD FEA	Crude MCS Mean	Crude MCS S.D.
(a) Maximum von Mises stress ($\sigma_{e,\max}$), MPa								
2	510.51	132.68	25	510.53	132.88	277	513.87	134.00
3	510.51	132.68	33	510.57	132.92	481		
(b) Maximum largest principal strain ($\epsilon_{1,\max}$), percent								
2	0.253	0.065	25	0.253	0.065	277	0.254	0.065
3	0.253	0.065	33	0.253	0.065	481		

Computational Cost

Univariate PDD (25-33 FEA); Bivariate PDD (277-481 FEA); Crude MCS (5000 FEA)

RESULTS

- **Probability Distributions of Stresses**



Computational Cost

Bivariate PDD (481 FEA); Crude MCS (5000 FEA)

CONCLUSIONS

- A new X-PDD method was developed for stochastic analysis of high-dimensional systems
  - Hierarchical decomposition with increasing dimensions
  - Fourier-polynomial expansions of component functions
  - Dimension-reduction integration for calculating coefficients
- No sample points were required; yet, generates convergent approximations
- Provides analytical formulae for statistical moments
- Predicts accurate tail probabilistic characteristics (reliability) at modest computational effort
- Computational complexity is polynomic; vitiates curse of dimensionality to some extent

FUTURE WORK

- Develop X-PDD method for reliability analysis of high-dimensional complex systems (Year 1)
  - ON basis & Fourier-polynomial expansions (completed)
  - Dimension-reduction integration for calculating coefficients (completed)
  - Enriched basis for non-smooth responses (ongoing)
- Integrate X-PDD and score functions for concurrent design sensitivity analysis (Year 2)
  - Formulation of score functions
  - Coupled score functions and X-PDD
- Develop fast and efficient reliability-based and robust design optimization algorithms (Year 3)
  - Special reliability-based design optimization
  - General reliability-based design optimization
  - Robust design optimization