A Comparison of Two Stochastic Expansion Methods for Solving Random Eigenvalue Problems

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Outline

Random Eigenvalue Problem

$$\lambda(\boldsymbol{X}) \in \mathcal{L}_2; \phi(\boldsymbol{X}) \in \mathcal{L}_2$$

$$\boldsymbol{X} \in \mathbb{R}^N \rightarrow \boxed{f\left(\lambda(\boldsymbol{X}); \mathbf{A}_1(\boldsymbol{X}), \cdots, \mathbf{A}_J(\boldsymbol{X})\right) \phi(\boldsymbol{X}) = \mathbf{0}} \rightarrow \begin{array}{l} \lambda(\boldsymbol{X}) \in \mathbb{R} \text{ or } \mathbb{C} \\ \phi(\boldsymbol{X}) \in \mathbb{R}^N \text{ or } \mathbb{C}^N \end{array}$$

$$\mathbf{A}_j \in \mathbb{R}^{N\times N}, j = 1, \ldots, J$$

Random eigenvalue problem	Problem type and application(s)
$\left[-\lambda(\boldsymbol{X})\boldsymbol{M}(\boldsymbol{X}) + \boldsymbol{K}(\boldsymbol{X})\right]\phi(\boldsymbol{X}) = \mathbf{0}$	*Linear*; undamped or proportionally damped systems
$\left[\lambda^2(\boldsymbol{X})\boldsymbol{M}(\boldsymbol{X}) + \lambda(\boldsymbol{X})\boldsymbol{C}(\boldsymbol{X}) + \boldsymbol{K}(\boldsymbol{X})\right]\phi(\boldsymbol{X}) = \mathbf{0}$	*Quadratic*; non − proportionally damped systems, singularity problems
$\left[\lambda(\boldsymbol{X})\boldsymbol{M}_1(\boldsymbol{X}) + \boldsymbol{M}_0(\boldsymbol{X}) + \boldsymbol{M}_1^T(\boldsymbol{X})/\lambda(\boldsymbol{X})\right]\phi(\boldsymbol{X}) = \mathbf{0}$	*Palindromic*; acoustic emissions in high − speed trains
$\left[\sum_k \lambda^k(\boldsymbol{X})\boldsymbol{A}_k(\boldsymbol{X})\right]\phi(\boldsymbol{X}) = \mathbf{0}$	*Polynomial*; control and dynamics problems
$\left[\lambda(\boldsymbol{X})\boldsymbol{M}(\boldsymbol{X}) - \boldsymbol{K}(\boldsymbol{X}) + \sum_k \frac{\lambda^q(\boldsymbol{X})\boldsymbol{C}_k(\boldsymbol{X})}{a_k - \lambda(\boldsymbol{X})}\right]\phi(\boldsymbol{X}) = \mathbf{0}$	*Rational*; plate vibration ($q = 1$), fluid− structure vibration ($q = 2$), vibration of viscoelastic materials

What are the probabilistic characteristics of random eigensolutions?

PDD - Polynomial Dimensional Decomposition (Rahman, 2008)

$$
\begin{aligned}
\lambda_{PDD}(\boldsymbol{X}) := \lambda_0 &+ \sum_{i=1}^{N}\sum_{j=1}^{\infty} C_{ij}\psi_{ij}(X_i) + \sum_{i_1<i_2}^{N}\sum_{j_2=1}^{\infty}\sum_{j_1=1}^{\infty} C_{i_1 i_2 j_1 j_2}\psi_{i_1 j_1}(X_{i_1})\psi_{i_2 j_2}(X_{i_2}) \\
&+ \sum_{i_1<i_2<i_3}^{N}\sum_{j_3=1}^{\infty}\sum_{j_2=1}^{\infty}\sum_{j_1=1}^{\infty} C_{i_1 i_2 i_3 j_1 j_2 j_3}\psi_{i_1 j_1}(X_{i_1})\psi_{i_2 j_2}(X_{i_2})\psi_{i_3 j_3}(X_{i_3}) \\
&+ \cdots + \sum_{i_1<i_2<\cdots<i_N}^{N}\sum_{j_N=1}^{\infty}\cdots\sum_{j_1=1}^{\infty} C_{i_1\cdots i_N j_1\cdots j_N}\prod_{q=1}^{N}\psi_{i_q j_q}(X_{i_q})
\end{aligned}
$$

$\lambda_0, C_{i_1\cdots i_s j_1\cdots j_s},\ s = 1,\cdots,N \quad \rightarrow \quad$ PDD expansion coefficients

$\psi_{i_q j_q}(X_{i_q}),\ q = 1,\cdots,N \quad \rightarrow \quad$ Univariate ON Polynomials

PCE - Polynomial Chaos Expansion (Wiener, 1938)

$$\boxed{\begin{aligned}\lambda_{PCE}(\boldsymbol{X}) := a_0\Gamma_0 &+ \sum_{i=1}^{N} a_i \Gamma_1(X_i) + \sum_{i_1=1}^{N}\sum_{i_2=i_1}^{N} a_{i_1 i_2}\Gamma_2(X_{i_1}, X_{i_2}) \\ &+ \sum_{i_1=1}^{N}\sum_{i_2=i_1}^{N}\sum_{i_3=i_2}^{N} a_{i_1 i_2 i_3}\Gamma_3(X_{i_1}, X_{i_2}, X_{i_3}) \\ &+ \cdots + \sum_{i_1=1}^{N}\cdots\sum_{i_p=i_{p-1}}^{N} a_{i_1\cdots i_p}\Gamma_p(X_{i_1}, \cdots, X_{i_p}) + \cdots\end{aligned}}$$

$$\begin{aligned}\Gamma_0 &= 1 \;;\; \Gamma_1(X_i) = \psi_{i1}(X_i) \\ \Gamma_2(X_{i_1}, X_{i_2}) &= \psi_{i_1 2}(X_{i_1})\delta_{i_1 i_2} - \psi_{i_1 1}(X_{i_1})\psi_{i_2 1}(X_{i_2})(\delta_{i_1 i_2} - 1) \\ \Gamma_3(X_{i_1}, X_{i_2}, X_{i_3}) &= \psi_{i_1 3}(X_{i_1})\delta_{i_1 i_2}\delta_{i_1 i_3}\delta_{i_2 i_3} \\ &- \psi_{i_1 1}(X_{i_1})\psi_{i_2 2}(X_{i_2})\delta_{i_2 i_3}(\delta_{i_1 i_2} - 1) \\ &- \psi_{i_2 2}(X_{i_2})\psi_{i_3 1}(X_{i_3})\delta_{i_1 i_2}(\delta_{i_2 i_3} - 1) \\ &- \psi_{i_1 1}(X_{i_1})\psi_{i_2 1}(X_{i_2})\psi_{i_3 1}(X_{i_3})(\delta_{i_1 i_2}\delta_{i_1 i_3}\delta_{i_2 i_3} - 1) \\ &\times (\delta_{i_1 i_2} - 1)(\delta_{i_2 i_3} - 1)\end{aligned}$$

Link between PDD and PCE

$$\lambda_{PCE}(\boldsymbol{X}) = \lambda_0 + \sum_{s=1}^{N}\left[\underbrace{\sum_{i_1=1}^{N-s+1}\cdots\sum_{i_s=i_{s-1}+1}^{N}}_{s\text{ sums}}\underbrace{\sum_{j_1=1}^{\infty}\cdot\cdot\sum_{j_s=1}^{\infty}}_{s\text{ sums}} C_{i_1\cdots i_s j_1\cdots j_s}\prod_{q=1}^{s}\psi_{i_q j_q}(X_{i_q})\right]$$
$$= \lambda_{PDD}(\boldsymbol{X})$$

$$\begin{aligned}
a_0 &= \lambda_0 \\
a_i &= C_{i1} \\
a_{i_1 i_2} &= C_{i_1 2}\delta_{i_1 i_2} - C_{i_1 i_2 11}(\delta_{i_1 i_2} - 1) \\
a_{i_1 i_2 i_3} &= C_{i_1 3}\delta_{i_1 i_2}\delta_{i_1 i_3}\delta_{i_2 i_3} - C_{i_1 i_2 12}\delta_{i_2 i_3}(\delta_{i_1 i_2} - 1) \\
&\quad - C_{i_2 i_3 21}\delta_{i_1 i_2}(\delta_{i_2 i_3} - 1) \\
&\quad - C_{i_1 i_2 i_3 111}(\delta_{i_1 i_2}\delta_{i_1 i_3}\delta_{i_2 i_3} - 1) \\
&\quad \times(\delta_{i_1 i_2} - 1)(\delta_{i_2 i_3} - 1) \\
\ldots &= \ldots,
\end{aligned}$$

Series Truncations and Approximations

- **PDD**

$$\tilde{\lambda}_{S,m}(\boldsymbol{X}) = \lambda_0 + \sum_{s=1}^{S}\left[\underbrace{\sum_{i_1=1}^{N-s+1}\cdots\sum_{i_s=i_{s-1}+1}^{N}}_{s\text{ sums}}\underbrace{\sum_{j_1=1}^{m}\cdots\sum_{j_s=1}^{m}}_{s\text{ sums}} C_{i_1\cdots i_s j_1\cdots j_s}\prod_{q=1}^{s}\psi_{i_q j_q}(X_{i_q})\right]$$

- **PCE**

$$\bar{\lambda}_{p}(\boldsymbol{X}) = \lambda_0 + \sum_{s=1}^{N}\left[\underbrace{\sum_{i_1=1}^{N-s+1}\cdots\sum_{i_s=i_{s-1}+1}^{N}}_{s\text{ sums}}\underbrace{\sum_{j_1=1}^{p-s+1}\cdots\sum_{j_s=1}^{p-s+1}}_{s\text{ sums};j_1+\cdots+j_s\le p} C_{i_1\cdots i_s j_1\cdots j_s}\prod_{q=1}^{s}\psi_{i_q j_q}(X_{i_q})\right]$$

For $S < N$, $p = m < \infty$; $\tilde{\lambda}_{S,m}(\boldsymbol{X}) \neq \bar{\lambda}_{p}(\boldsymbol{X})$

Which one is better ?

Variance Approximations

- **PDD**

$$\mathbb{E}\left[\tilde{\lambda}_{S,m}(\boldsymbol{X}) - \lambda_0\right]^2 = \sum_{s=1}^{S}\left(\underbrace{\sum_{i_1=1}^{N-s+1} \cdots \sum_{i_s=i_{s-1}+1}^{N}}_{s \text{ sums}} \quad \underbrace{\sum_{j_1=1}^{m} \cdots \sum_{j_s=1}^{m}}_{s \text{ sums}} C^2_{i_1\cdots i_s j_1 \cdots j_s} \right)$$

- **PCE**

$$\mathbb{E}\left[\bar{\lambda}_{p}(\boldsymbol{X}) - \lambda_0\right]^2 = \sum_{s=1}^{N}\left(\underbrace{\sum_{i_1=1}^{N-s+1} \cdots \sum_{i_s=i_{s-1}+1}^{N}}_{s \text{ sums}} \quad \underbrace{\sum_{j_1=1}^{p-s+1} \cdots \sum_{j_s=1}^{p-s+1}}_{s \text{ sums}; j_1+\cdots+j_s \le p} C^2_{i_1\cdots i_s j_1 \cdots j_s} \right)$$

$$C_{i_1\cdots i_s j_1\cdots j_s} := \int_{\mathbb{R}^N} \lambda(\boldsymbol{x}) \prod_{q=1}^{s} \psi_{i_q j_q}(x_{i_q}) f_{\boldsymbol{X}}(\boldsymbol{x})\, d\boldsymbol{x}$$

Dimension-Reduction Integration (Xu and Rahman 2004)

$$\hat{y}_R(\boldsymbol{x}) = \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \sum_{i_1,\cdots,i_{R-k}=1;i_1<\cdots<i_{R-k}}^{N} \times y(c_1,\cdots,c_{i_1-1},x_{i_1},c_{i_1+1},\cdots,c_{i_{R-k}-1},x_{i_{R-k}},c_{i_{R-k}+1},\cdots,c_N)$$

$$C_{i_1\cdots i_s j_1\cdots j_s} \cong \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \sum_{i_1,\cdots,i_{R-k}=1;i_1<\cdots<i_{R-k}}^{N} \times \int_{\mathbb{R}^{R-k}} y(c_1,\cdots,c_{i_1-1},x_{i_1},c_{i_1+1},\cdots,c_{i_{R-k}-1},x_{i_{R-k}}, c_{i_{R-k}+1},\cdots,c_N) \prod_{p=1}^{s}\psi_{i_p j_p}(x_{i_p}) \prod_{p=1}^{R-k} f_{k_p}(x_{k_p})\, dx_{k_p}$$

Dimension-Reduction Integration (Xu and Rahman 2004)

$$\hat{y}_R(\boldsymbol{x}) = \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \sum_{i_1,\cdots,i_{R-k}=1; i_1<\cdots<i_{R-k}}^{N} \times y(c_1,\cdots,c_{i_1-1},x_{i_1},c_{i_1+1},\cdots,c_{i_{R-k}-1},x_{i_{R-k}},c_{i_{R-k}+1},\cdots,c_N)$$

$$C_{i_1\cdots i_s j_1\cdots j_s} \cong \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \sum_{i_1,\cdots,i_{R-k}=1; i_1<\cdots<i_{R-k}}^{N} \times \int_{\mathbb{R}^{R-k}} y(c_1,\cdots,c_{i_1-1},x_{i_1},c_{i_1+1},\cdots,c_{i_{R-k}-1},x_{i_{R-k}}, c_{i_{R-k}+1},\cdots,c_N) \prod_{p=1}^{s}\psi_{i_p j_p}(x_{i_p}) \prod_{p=1}^{R-k} f_{k_p}(x_{k_p})\,dx_{k_p}$$

- Highly efficient when $R = S \ll N$ (also convergent)
- Polynomial complexity; 1D or 2D integration for univariate or bivariate approximation

Error Analysis

$$\text{PDD error}: e_{S,m} := \mathbb{E}\left[\lambda(\boldsymbol{X}) - \tilde{\lambda}_{S,m}(\boldsymbol{X})\right]^2 = \mathbb{E}\left[\lambda_{PDD}(\boldsymbol{X}) - \tilde{\lambda}_{S,m}(\boldsymbol{X})\right]^2$$

$$\text{PCE error}: e_p := \mathbb{E}\left[\lambda(\boldsymbol{X}) - \bar{\lambda}_p(\boldsymbol{X})\right]^2 = \mathbb{E}\left[\lambda_{PDD}(\boldsymbol{X}) - \bar{\lambda}_p(\boldsymbol{X})\right]^2$$

$$\Rightarrow e_{S,m} = \sum_{s=1}^{S}\left(\sum_{i_1=1}^{N-s+1}\cdots\sum_{i_s=i_{s-1}+1}^{N}\sum_{j_1=m+1}^{\infty}\cdots\sum_{j_s=m+1}^{\infty} C^2_{i_1\cdots i_s j_1\cdots j_s}\right) + \sum_{s=S+1}^{N}\left(\sum_{i_1=1}^{N-s+1}\cdots\sum_{i_s=i_{s-1}+1}^{N}\sum_{j_1=1}^{\infty}\cdots\sum_{j_s=1}^{\infty} C^2_{i_1\cdots i_s j_1\cdots j_s}\right)$$

$$\Rightarrow e_p = \sum_{s=1}^{N}\left(\sum_{i_1=1}^{N-s+1}\cdots\sum_{i_s=i_{s-1}+1}^{N}\sum_{\substack{j_1=1 \\ j_1+\cdots+j_s\le p}}^{\infty}\cdots\sum_{j_s=1}^{\infty} C^2_{i_1\cdots i_s j_1\cdots j_s}\right)$$

For $p = m$; $C_{i_1\cdots i_s j_1\cdots j_s} = 0$; $s = S+1, \ldots, N$,

$$\boxed{e_m \geq e_{S,m}}$$

Example 1: A 2-DoF, Undamped, Spring-Mass System

Random Input

- Spring Stiffnesses

$$\begin{aligned} K_1(\mathbf{X}) &= 1000 X_1 \text{ N/m} \\ K_2(\mathbf{X}) &= 1100 X_2 \text{ N/m} \\ K_3(\mathbf{X}) &= 100 X_3 \text{ N/m} \end{aligned}$$

- Masses

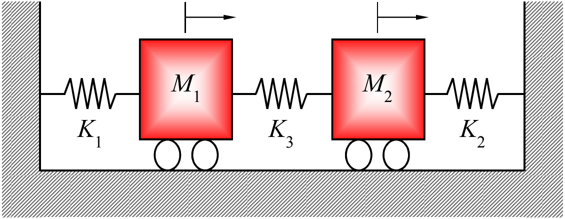
$$\begin{aligned} M_1(\mathbf{X}) &= X_4 \text{ kg} \\ M_2(\mathbf{X}) &= 1.5 X_5 \text{ kg} \end{aligned}$$

$$\boldsymbol{X} = \{X_1, \cdots, X_5\}^T \in \mathbb{R}^5$$

$$\boldsymbol{\mu}_{\mathbf{X}} = \mathbf{1} \in \mathbb{R}^5$$

$$v_1 = v_2 = 0.25$$

$$v_3 = v_4 = v_5 = 0.125$$

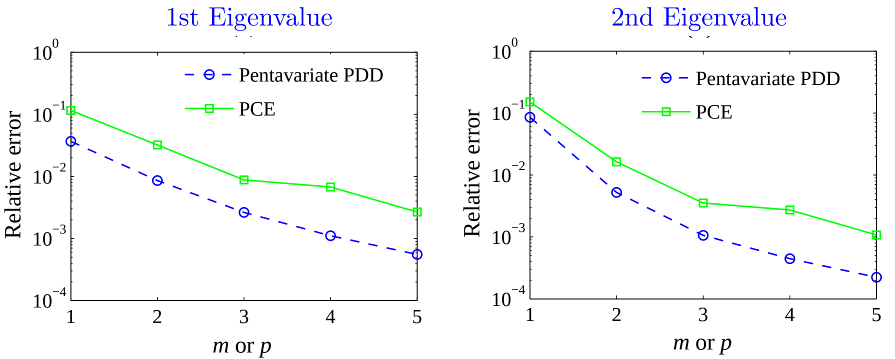


$$\boldsymbol{M}(\boldsymbol{X}) = \begin{bmatrix} M_1(\boldsymbol{X}) & 0 \\ 0 & M_2(\boldsymbol{X}) \end{bmatrix}$$

$$\boldsymbol{K}(\boldsymbol{X}) = \begin{bmatrix} K_1(\boldsymbol{X}) + K_3(\boldsymbol{X}) & -K_3(\boldsymbol{X}) \\ -K_3(\boldsymbol{X}) & K_2(\boldsymbol{X}) + K_3(\boldsymbol{X}) \end{bmatrix}$$

$$\begin{aligned} \mathbf{X} &\sim LN\left(\boldsymbol{\mu}_X, \boldsymbol{\Sigma}_X\right) \\ \boldsymbol{\Sigma}_X &= \text{diag}(v_1^2, v_2^2, v_3^2, v_4^2, v_5^2) \in \mathbb{R}^{5\times 5} \end{aligned}$$

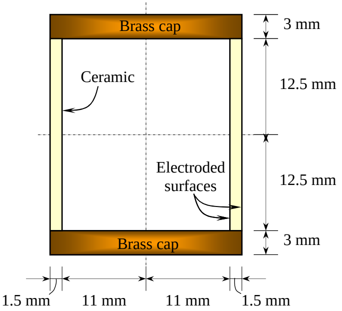
Example 1: Errors in Variances of Eigenvalues

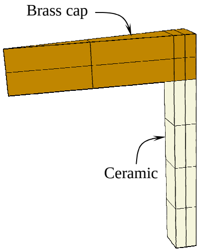


Second-Moment Errors

PDD commits smaller error than does PCE for identical expansion orders

Example 2: A Piezoelectric Transducer





Statistical Prop. of Input ($\boldsymbol{X} \in \mathbb{R}^{14}$)

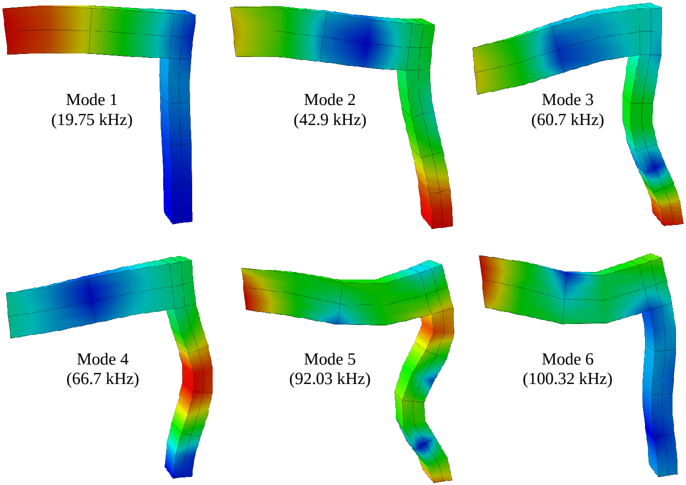
RV	Property[(a)]	Mean	COV
X_1, GPa	D_{1111}	115.4	0.15
X_2, GPa	D_{1122}, D_{1133}	74.28	0.15
X_3, GPa	D_{2222}, D_{3333}	139	0.15
X_4, GPa	D_{2233}	77.84	0.15
X_5, GPa	$D_{1212}, D_{2323}, D_{1313}$	25.64	0.15
X_6, C/m^2	e_{111}	15.08	0.1
X_7, C/m^2	e_{122}, e_{133}	-5.207	0.1
X_8, C/m^2	e_{212}, e_{313}	12.71	0.1
X_9, nF/m	D_{11}	5.872	0.1
X_{10}, nF/m	D_{22}, D_{33}	6.752	0.1
X_{11}, GPa	E_b	104	0.15
X_{12}	ν_b	0.37	0.05
X_{13}, g/m^3	ρ_b	8500	0.15
X_{14}, g/m^3	ρ_c	7500	0.15

(a) D_{ijkl}, D_{ij}, e_{ijk} = elastic moduli, dielectric, piezoelectric constants of ceramic; E_b, ν_b, ρ_b = elastic modulus, Poisson's ratio, and density of brass; ρ_c = density of ceramic

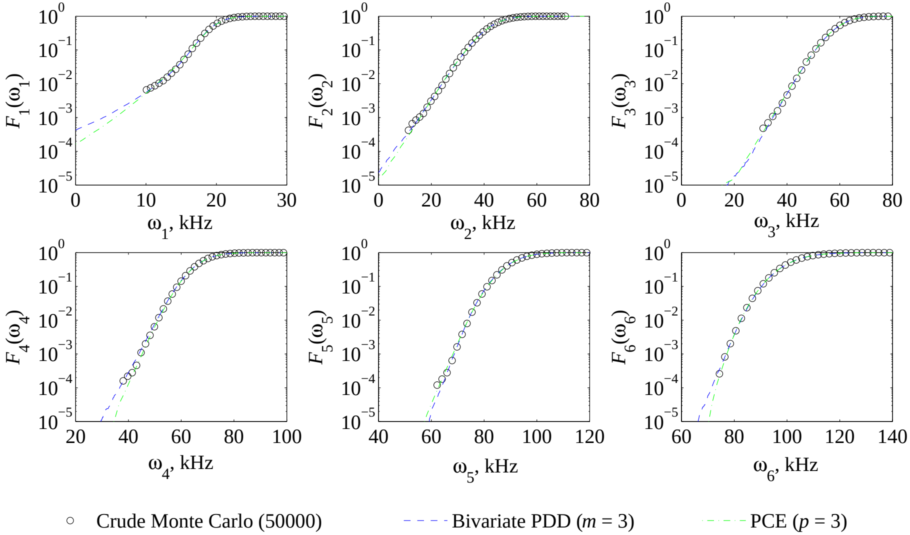
$S = 2, m = 3, n = 4, N = 14, L = 10^6$

All coeffs. calculated by S-dim.-red. integ.

Example 2:First Six Mode Shapes (Mean Input)



Example 2: Marginal Distributions of Frequencies



Bivariate PDD: 1,513 FEA PCE: 24,809 FEA

Conclusions

- Two polynomial expansion methods, PDD and PCE, investigated
- Infinite series of PDD and PCE are the same; not so for their truncations
- PCE approximation expressed in terms of PDD expansion coefficients
- For equal expansion orders, error in variance from PDD will never be greater than that from PCE
- PDD can be significantly more efficient than PCE

Rahman, S. and Yadav, V., Orthogonal Polynomial Expansions for Solving Random Eigenvalue Problems, *Int. J. Uncertainty Quantification*, 1(2): 163-187 (2011).