Robust Design Optimization by Polynomial Dimensional Decomposition

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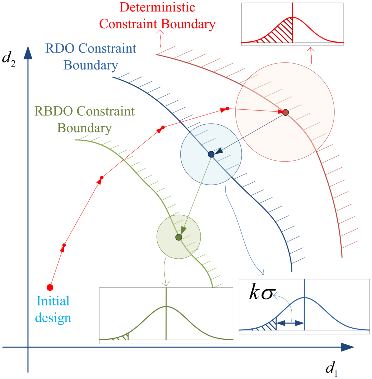
Outline

Design Under Uncertainty

Input $\mathbf{X} \in \mathbb{R}^N \rightarrow$ **COMPLEX SYSTEM** $\rightarrow$ Output $y_l(\mathbf{X}) \in \mathbb{R}$

$$\mathbf{X} = (X_1, \cdots X_N) \in \mathbb{R}^N \sim f_{\mathbf{X}}(\mathbf{x};\mathbf{d}) \rightarrow \text{random variables}$$

$$\mathbf{d} = (d_1, \cdots d_M) \in \mathcal{D} \subseteq \mathbb{R}^M \rightarrow \text{design parameters}$$



RDO

- **Problem Definition**

$$\min_{\mathbf{d}\in\mathcal{D}\subseteq\mathbb{R}^M} \quad c_0(\mathbf{d}) := w_1 \mathbb{E}_{\mathbf{d}}\left[y_0(\mathbf{X})\right]/\mu_0^* + w_2\sqrt{\mathrm{var}_{\mathbf{d}}\left[y_0(\mathbf{X})\right]}/\sigma_0^*,$$

$$\text{subject to} \quad c_l(\mathbf{d}) := \alpha_l\sqrt{\mathrm{var}_{\mathbf{d}}\left[y_l(\mathbf{X})\right]} - \mathbb{E}_{\mathbf{d}}\left[y_l(\mathbf{X})\right] \leq 0;\ l = 1,\cdots,K,$$

$$d_{k,L} \leq d_k \leq d_{k,U},\ k = 1,\cdots,M$$

- **Existing Methods for RDO**
  - Taylor Series Expansion
  - Point Estimate Methods
  - Polynomial Chaos Expansion (PCE)
  - Tensor Product Quadrature (TPQ)
  - Others

Polynomial Dimensional Decomposition(Rahman, 2008)

Input $\mathbf{X} \in \mathbb{R}^N \rightarrow$ **COMPLEX SYSTEM** $\rightarrow$ Output $y(\mathbf{X}) \in \mathcal{L}_2(\mathbb{R})$

$X_i \sim f_{X_i}(x_i; \mathbf{d})$; indep.; $\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \mathbf{d}) = \prod_{p=1}^{|u|} \psi_{i_p j_p}(X_{i_p}; \mathbf{d})$

- **Polynomial Dimensional Decomposition (PDD)**

$$y(\mathbf{X}) = y_\emptyset(\mathbf{d}) + \sum_{\emptyset \neq u \subseteq \{1,\cdots,N\}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|} \\ j_1,\cdots,j_{|u|} \neq 0}} C_{u\mathbf{j}_{|u|}}(\mathbf{d}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \mathbf{d})$$

- **S-variate, mth-order PDD Approximation**

$$\tilde{y}_{S,m}(\mathbf{X}) = y_\emptyset(\mathbf{d}) + \sum_{\substack{\emptyset \neq u \subseteq \{1,\cdots,N\} \\ 1 \leq |u| \leq S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, ||\mathbf{j}_{|u|}||_\infty \leq m \\ j_1,\cdots,j_{|u|} \neq 0}} C_{u\mathbf{j}_{|u|}}(\mathbf{d}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u; \mathbf{d})$$

$y_\emptyset(\mathbf{d}) := \int_{\mathbb{R}^N} y(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x}; \mathbf{d}) d\mathbf{x}$, $C_{u\mathbf{j}_{|u|}}(\mathbf{d}) := \int_{\mathbb{R}^N} y(\mathbf{x}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u; \mathbf{d}) f_{\mathbf{X}}(\mathbf{x}; \mathbf{d}) d\mathbf{x}$

Statistical Moments

- **Two Important Properties of Polynomial Basis**

$$\mathbb{E}_{\mathbf{d}}\left[\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})\right] = 0$$

$$\mathbb{E}_{\mathbf{d}}\left[\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u;\mathbf{d})\psi_{v\mathbf{j}_{|v|}}(\mathbf{X}_u;\mathbf{d})\right] = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{if } u \neq v. \end{cases}$$

- **Second-Moment Statistics**

$$\mathbb{E}_{\mathbf{d}}\left[\tilde{y}_{S,m}(\mathbf{X})\right] = y_\emptyset(\mathbf{d})$$

$$\text{var}_{\mathbf{d}}[\tilde{y}_{S,m}(\mathbf{X})] = \sum_{\substack{\emptyset \neq u \subseteq \{1,\cdots,N\} \\ 1 \leq |u| \leq S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, ||\mathbf{j}_{|u|}||_\infty \leq m \\ j_1,\cdots,j_{|u|} \neq 0}} C^2_{u\mathbf{j}_{|u|}}(\mathbf{d})$$

Sensitivities of Statistical Moments

- **Score Functions**

$$\frac{\partial \mathbb{E}_{\mathbf{d}}\left[y^r(\mathbf{X})\right]}{\partial d_k} = \int_{\mathbb{R}^N} y^r(\mathbf{x}) \overbrace{\frac{\partial \ln f_{X_{i_k}}(x_{i_k};\mathbf{d})}{\partial d_k}}^{:= s_k(x_{i_k};\mathbf{d})} f_{\mathbf{X}}(\mathbf{x};\mathbf{d})\, d\mathbf{x}$$
$$:= \mathbb{E}_{\mathbf{d}}\left[y^r(\mathbf{X}) s_k(X_{i_k};\mathbf{d})\right]$$

$$s_k(X_{i_k};\mathbf{d}) \approx s_{k,\emptyset}(\mathbf{d}) + \sum_{j=1}^{m'} D_{i_k,j}(\mathbf{d}) \psi_{i_k j}(X_{i_k};\mathbf{d})$$

- **Design Sensitivities**

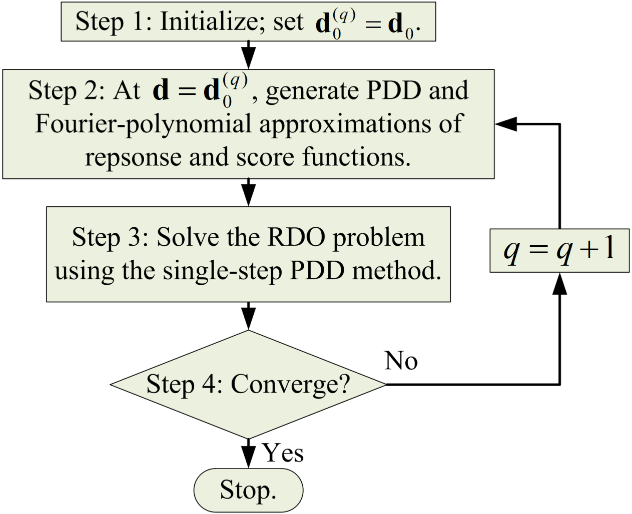
$$\frac{\partial \mathbb{E}_{\mathbf{d}}\left[\tilde{y}_{S,m}(\mathbf{X})\right]}{\partial d_k} = s_{k,\emptyset}(\mathbf{d}) y_{\emptyset}(\mathbf{d}) + \sum_{j=1}^{m_{\min}} C_{i_k j}(\mathbf{d}) D_{i_k,j}(\mathbf{d})$$

$$\frac{\partial \mathbb{E}_{\mathbf{d}}\left[\tilde{y}^2_{S,m}(\mathbf{X})\right]}{\partial d_k} = s_{k,\emptyset}(\mathbf{d}) y^2_{\emptyset}(\mathbf{d}) + 2 y_{\emptyset}(\mathbf{d}) \sum_{j=1}^{m_{\min}} C_{i_k j}(\mathbf{d}) D_{i_k,j}(\mathbf{d}) + s_{k,\emptyset}(\mathbf{d}) \mathrm{var}_{\mathbf{d}}\left[\tilde{y}_{S,m}(\mathbf{X})\right] + \tilde{T}_{k,m_{\min}}$$

Four New Methods

- **Direct PDD (Global)**
  - Straightforward integration of PDD with gradient-based optimization algorithms
  - Re-calculation of the PDD expansion coefficients
- **Single-Step PDD (Global)**
  - Single stochastic analysis by recycling PDD coefficients
  - Premature design solutions for practical problems
- **Sequential PDD (Global)**
  - Combination of single-step and direct-PDD
  - More expensive than single-step PDD, but substantially more economical than direct PDD
- **Multipoint Single-Step PDD (Local)**
  - A succession of simpler RDO sub-problems
  - Solution of practical problems using low-order and/or low-variate PDD approximations

Sequential PDD



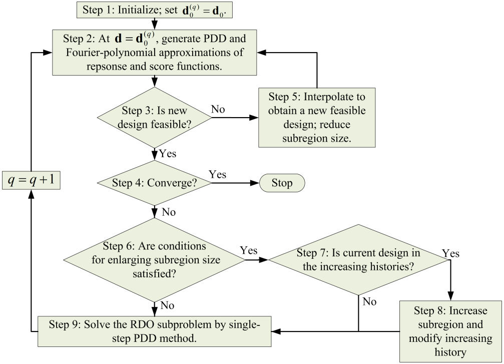
Multi-Point Single-Step PDD

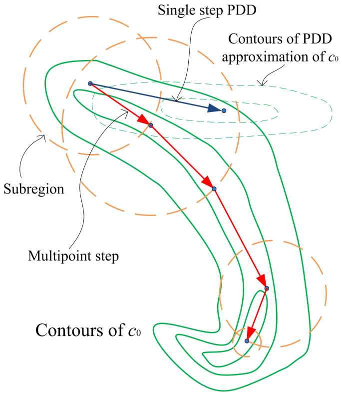
- **The RDO Subproblem**

$$\min_{\mathbf{d}\in\mathcal{D}^{(q)}\subseteq\mathcal{D}} \quad \tilde{c}^{(q)}_{0,S,m}(\mathbf{d}) := w_1 \frac{\mathbb{E}_{\mathbf{d}}\left[\tilde{y}^{(q)}_{0,S,m}(\mathbf{X})\right]}{\mu_0^*} + w_2 \frac{\sqrt{\operatorname{var}_{\mathbf{d}}\left[\tilde{y}^{(q)}_{0,S,m}(\mathbf{X})\right]}}{\sigma_0^*},$$

$$\text{subject to} \quad \tilde{c}^{(q)}_{l,S,m}(\mathbf{d}) = \alpha_l \sqrt{\operatorname{var}_{\mathbf{d}}\left[\tilde{y}^{(q)}_{l,S,m}(\mathbf{X})\right]} - \mathbb{E}_{\mathbf{d}}\left[\tilde{y}^{(q)}_{l,S,m}(\mathbf{X})\right] \le 0,\; l = 1,\cdots,K,$$

$$d^{(q)}_{k,0} - \beta^{(q)}_k (d_{k,U} - d_{k,L})/2 \le d_k \le d^{(q)}_{k,0} + \beta^{(q)}_k (d_{k,U} - d_{k,L})/2,\; k = 1,\cdots,M.$$



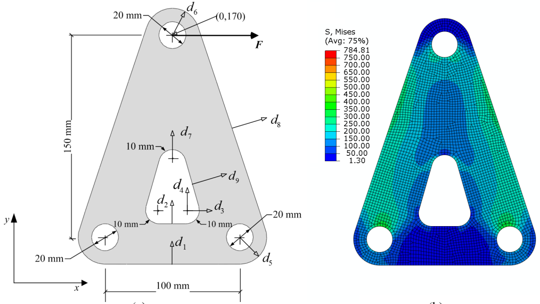


Optimization of a Mathematical Function

$$
\begin{aligned}
\min_{\mathbf{d}\in\mathcal{D}} \quad & c_0(\mathbf{d}) = \frac{\sqrt{\mathrm{var}_{\mathbf{d}}\left[y_0(\mathbf{X})\right]}}{15}, \\
\text{subject to} \quad & c_1(\mathbf{d}) = 3\sqrt{\mathrm{var}_{\mathbf{d}}\left[y_1(\mathbf{X})\right]} - \mathbb{E}_{\mathbf{d}}\left[y_1(\mathbf{X})\right] \leq 0, \\
& 1 \leq d_1 \leq 10,\ 1 \leq d_2 \leq 10, \\
& y_0(\mathbf{X}) = (X_1-4)^3 + (X_1-3)^4 + (X_2-5)^2 + 10 \\
& y_1(\mathbf{X}) = X_1 + X_2 - 6.45 \\
& X_i \sim \text{Gaussian variables; } d_i = \mathbb{E}[X_i]
\end{aligned}
$$

Results	Direct PDD	Single-Step PDD	TPQ
$\tilde{d}_1^*$	3.3508	3.3508	3.4449
$\tilde{d}_1^*$	4.9856	4.9856	5.000
$c_0(\tilde{\mathbf{d}}^*)$	0.0756	0.0756	0.0861
$c_1(\tilde{\mathbf{d}}^*)$	-0.1873	-0.1599	-0.2978
No. of iterations	5	5	4
No. of y_0 evaluations	66	11	81
No. of y_1 evaluations	30	5	81

Shape Optimization of a Three-Hole Bracket



(a) (b)

$$\min_{\mathbf{d}\in\mathcal{D}} \quad c_0(\mathbf{d}) = 0.5\frac{\mathbb{E}_{\mathbf{d}}\left[y_0(\mathbf{X})\right]}{\mathbb{E}_{\mathbf{d}_0}\left[y_0(\mathbf{X})\right]} + 0.5\frac{\sqrt{\mathrm{var}_{\mathbf{d}}\left[y_0(\mathbf{X})\right]}}{\sqrt{\mathrm{var}_{\mathbf{d}_0}\left[y_0(\mathbf{X})\right]}},$$

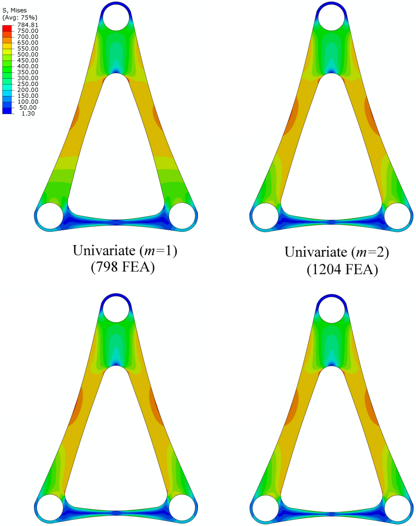
$$\text{subject to} \quad c_1(\mathbf{d}) = 3\sqrt{\mathrm{var}_{\mathbf{d}}\left[y_1(\mathbf{X})\right]} - \mathbb{E}_{\mathbf{d}}\left[y_1(\mathbf{X})\right] \leq 0$$

$$y_0(\mathbf{X}) = \rho\int_{\mathcal{D}'(\mathbf{X})} d\mathcal{D}';\ \ y_1(\mathbf{X}) = S_y - \sigma_{e,\max}(\mathbf{X})$$

$$X_i \sim \text{truncated Gaussian variables};\ d_i = \mathbb{E}[X_i]$$

Results

- **Optimal Bracket Designs**



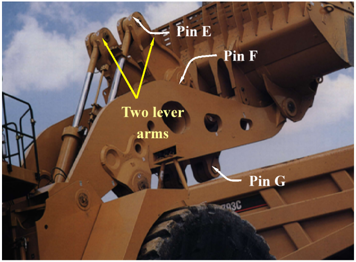
Univariate (m=1) (798 FEA)

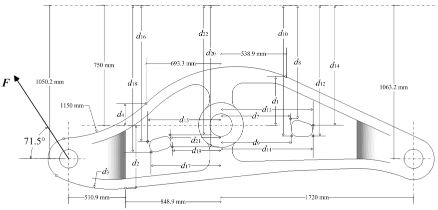
Univariate (m=2) (1204 FEA)

Univariate (m=3) (1332 FEA)

Bivariate (m=1) (6357 FEA)

Shape Optimization of a Lever-Arm





$$\min_{\mathbf{d}\in\mathcal{D}} \quad c_0(\mathbf{d}) = 0.5\frac{\mathbb{E}_{\mathbf{d}}\left[y_0(\mathbf{X})\right]}{\mathbb{E}_{\mathbf{d}_0}\left[y_0(\mathbf{X})\right]} + 0.5\frac{\sqrt{\mathrm{var}_{\mathbf{d}}\left[y_0(\mathbf{X})\right]}}{\sqrt{\mathrm{var}_{\mathbf{d}_0}\left[y_0(\mathbf{X})\right]}},$$

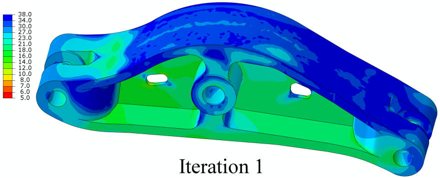
$$\text{subject to} \quad c_1(\mathbf{d}) = 3\sqrt{\mathrm{var}_{\mathbf{d}}\left[y_1(\mathbf{X})\right]} - \mathbb{E}_{\mathbf{d}}\left[y_1(\mathbf{X})\right] \leq 0$$

$$y_0(\mathbf{X}) = \rho\int_{\mathcal{D}'(\mathbf{X})} d\mathcal{D}'; \;\; y_1(\mathbf{X}) = N_{\min}(\mathbf{X}) - N_c$$

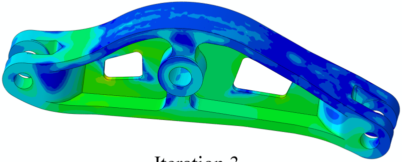
$$X_i \sim \text{truncated Gaussian variables; } d_i = \mathbb{E}[X_i]$$

Results

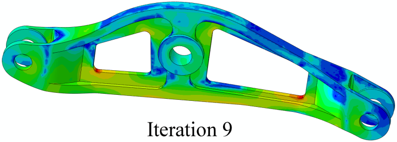
- **Fatigue Life Contours at Design Iterations**



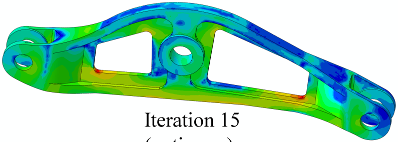
Iteration 1



Iteration 3



Iteration 9



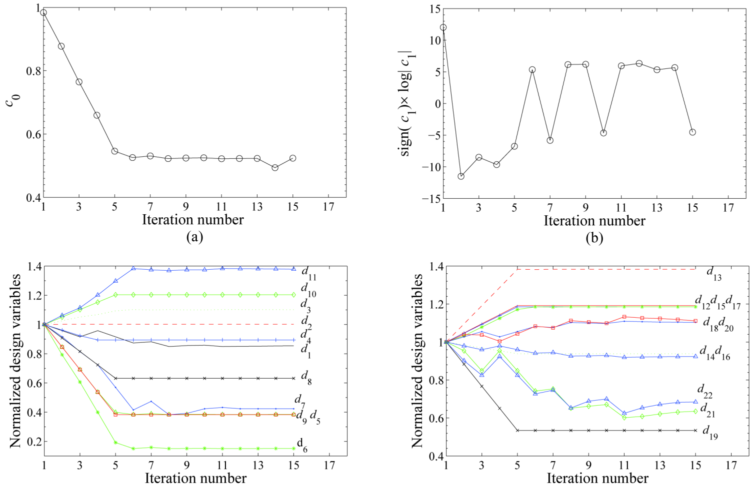
Iteration 15 (optimum)

Summary

- Optimal mass: 1263 kg (79% reduction of initial mass)
- Required 15 iterations and 675 FEA

Results

- **RDO Design Histories**



(c)

Conclusions and Future Work

Conclusions

- Four new methods for RDO of complex systems
- Novel integration of PDD and score function for calculating probabilistic responses and sensitivities simultaneously
- Capable of solving practical problems using low-order or low variate PDD approximations

Future Work

- Integration of PDD and subset simulation for calculating failure probability
- Component- and system-level RBDO