Multiplicative Polynomial Dimensional Decompositions for Uncertainty Quantification of High-Dimensional Complex Systems

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Outline

ANOVA Dimensional Decomposition

Input $\mathbf{X} \in \mathbb{R}^N \rightarrow$ **COMPLEX SYSTEM** $\rightarrow$ Output $y(\mathbf{X}) \in \mathcal{L}_2(\mathbb{R})$

$$
\begin{aligned}
y(\mathbf{X}) &= y_0 + \sum_{i=1}^{N} y_i(X_i) + \sum_{i_1,i_2=1;i_1<i_2}^{N} y_{i_1 i_2}(X_{i_1}, X_{i_2}) + \cdots + \\
&\quad \sum_{i_1,\cdots,i_s=1, i_1<\cdots<i_s}^{N} y_{i_1\cdots i_s}(X_{i_1}, \cdots, X_{i_s}) + \cdots + y_{12\cdots N}(X_1, \cdots, X_N)
\end{aligned}
$$

- **Compact Form**

$$
\begin{aligned}
y(\mathbf{X}) &= \sum_{u\subseteq\{1,\cdots,N\}} y_u(\mathbf{X}_u), \\
y_\emptyset &:= \int_{\mathbb{R}^N} y(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}, \\
y_u(\mathbf{X}_u) &:= \int_{\mathbb{R}^{N-|u|}} y(\mathbf{X}_u, \mathbf{x}_{-u}) f_{\mathbf{X}_{-u}}(\mathbf{x}_{-u})\, d\mathbf{x}_{-u} \\
&\quad - \sum_{v\subset u} y_v(\mathbf{X}_v).
\end{aligned}
$$

- **Orthogonality**

$$
\begin{aligned}
\mathbb{E}\left[y_u(\mathbf{X}_u)\right] &= 0; \\
\mathbb{E}\left[y_u(\mathbf{X}_u) y_v(\mathbf{X}_v)\right] &= \begin{cases} 0, & u \neq v \\ 1, & u = v \end{cases}
\end{aligned}
$$

Additive PDD (A-PDD) (Rahman, 2008)

$$y_{PDD}(\mathbf{X}) := y_\emptyset + \sum_{\emptyset \neq u \subseteq \{1,\cdots,N\}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|} \\ j_1,\cdots,j_{|u|} \neq 0}} C_{u\mathbf{j}_{|u|}} \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)$$

$$\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u) := \prod_{p=1}^{|u|} \psi_{i_p j_p}(X_{i_p}) \rightarrow \text{Univariate ON Polynomials}$$

$$y_\emptyset := \int_{\mathbb{R}^N} y(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}; \quad C_{u\mathbf{j}_{|u|}} := \int_{\mathbb{R}^N} y(\mathbf{x}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

- **S-variate, m-th order A-PDD Approximation**

$$\tilde{y}_{S,m}(\mathbf{X}) = y_\emptyset + \sum_{\substack{\emptyset \neq u \subseteq \{1,\cdots,N\} \\ 1 \leq |u| \leq S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \|\mathbf{j}_{|u|}\|_\infty \leq m \\ j_1,\cdots,j_{|u|} \neq 0}} C_{u\mathbf{j}_{|u|}} \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)$$

Suitable only for systems with additive dimensional hierarchy

Multiplicative Dimensional Decomposition

- **Long Form (F-HDMR; Tunga & Demiralp, 2005)**

$$y(\mathbf{X}) = (1+z_0)\left[\prod_{i=1}^{N}\{1+z_i(X_i)\}\right]\left[\prod_{i_1<i_2}^{N}\{1+z_{i_1 i_2}(X_{i_1},X_{i_2})\}\right] \times\cdots\times[1+z_{12\cdots N}(X_1,\cdots X_N)]$$

- **Compact Form**

$$y(\mathbf{X}) = \prod_{u\subseteq\{1,\cdots,N\}}^{N}[1+z_u(\mathbf{X}_u)]$$

$$1+z_u(\mathbf{X}_u) := ?$$

The decomposition exists and is unique.

Multiplicative Dimensional Decomposition

- **Link with ANOVA**

$$1 + z_u(\mathbf{X}_u) = \frac{\sum_{v \subseteq u} y_v(\mathbf{X}_v)}{\prod_{v \subset u} [1 + z_v(\mathbf{X}_v)]}$$

- **Special Cases**

$$1 + z_\emptyset = y_\emptyset$$

$$1 + z_{\{i\}}(X_i) = \frac{y_\emptyset + y_{\{i\}}(X_i)}{y_\emptyset}$$

$$1 + z_{\{i_1,i_2\}}(X_{i_1}, X_{i_2}) = \frac{y_\emptyset + y_{\{i_1\}}(X_{i_1}) + y_{\{i_2\}}(X_{i_2}) + y_{\{i_1,i_2\}}(X_{i_1}, X_{i_2})}{y_\emptyset \left[\frac{y_\emptyset + y_{\{i_1\}}(X_{i_1})}{y_\emptyset}\right]\left[\frac{y_\emptyset + y_{\{i_2\}}(X_{i_2})}{y_\emptyset}\right]}$$

Factorized PDD (F-PDD)

- **Orthogonal Polynomial Expansion**

$$y_u(\mathbf{X}_u) = \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|} \\ j_1,\cdots,j_{|u|} \neq 0}} C_{u\mathbf{j}_{|u|}} \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u), \; \emptyset \neq u \subseteq \{1,\cdots,N\}$$

- **Exact**

$$y(\mathbf{X}) = y_\emptyset \prod_{\emptyset \neq u \subseteq \{1,\cdots,N\}} \left[\frac{y_\emptyset + \sum\limits_{\emptyset \neq v \subseteq u} \sum\limits_{\substack{\mathbf{j}_{|v|} \in \mathbb{N}_0^{|v|} \\ j_1,\cdots,j_{|v|} \neq 0}} C_{v\mathbf{j}_{|v|}} \psi_{v\mathbf{j}_{|v|}}(\mathbf{X}_v)}{\prod\limits_{v \subset u} [1 + z_v(\mathbf{X}_v)]} \right]$$

- **Approximate**

$$\hat{y}_{S,m}(\mathbf{X}) = y_\emptyset \prod_{\substack{\emptyset \neq u \subseteq \{1,\cdots,N\} \\ 1 \leq |u| \leq S}} \left[\frac{y_\emptyset + \sum\limits_{\emptyset \neq v \subseteq u} \sum\limits_{\substack{\mathbf{j}_{|v|} \in \mathbb{N}_0^{|v|}, \|\mathbf{j}_{|u|}\|_\infty \leq m \\ j_1,\cdots,j_{|v|} \neq 0}} C_{v\mathbf{j}_{|v|}} \psi_{v\mathbf{j}_{|v|}}(\mathbf{X}_v)}{\prod\limits_{v \subset u} [1 + z_v(\mathbf{X}_v)]} \right]$$

$$\hat{y}_{S,m}(\mathbf{X}) \neq \tilde{y}_{S,m}(\mathbf{X})$$

F-PDD

- **Univariate ($S = 1$), mth-order F-PDD Approx.**

$$\hat{y}_{1,m}(\mathbf{X}) = y_\emptyset \left[\prod_{i=1}^{N} \left\{ 1 + \frac{1}{y_\emptyset} \sum_{j=1}^{m} C_{ij} \psi_{ij}(X_i) \right\} \right]$$

- **Statistical Moments**

$$\mathbb{E}\left[\hat{y}_{1,m}(\mathbf{X})\right] = y_\emptyset$$

$$\mathbb{E}\left[\left(\hat{y}_{1,m}(\mathbf{X}) - \mathbb{E}\left[\hat{y}_{1,m}(\mathbf{X})\right]\right)^2\right] = y_\emptyset^2 \left[\prod_{i=1}^{N} \left(1 + \frac{1}{y_\emptyset^2} \sum_{j=1}^{m} C_{ij}^2 \right) - 1 \right]$$

Logarithmic PDD (L-PDD)

- **ANOVA Decomposition of $w(\mathbf{X}) := \ln y(\mathbf{X})$**

$$
\begin{aligned}
w(\mathbf{X}) &= \sum_{u\subseteq\{1,\cdots,N\}} w_u(\mathbf{X}_u),\\
w_\emptyset &:= \int_{\mathbb{R}^N} \ln y(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x},\\
w_u(\mathbf{X}_u) &:= \int_{\mathbb{R}^{N-|u|}} \ln y(\mathbf{X}_u,\mathbf{x}_{-u}) f_{\mathbf{X}_{-u}}(\mathbf{x}_{-u})\,d\mathbf{x}_{-u} - \sum_{v\subset u} w_v(\mathbf{X}_v)
\end{aligned}
$$

- **Inversion**

$$
y(\mathbf{X}) = \prod_{u\subseteq\{1,\cdots,N\}} \exp\left[w_u(\mathbf{X}_u)\right]
$$

L-PDD

- **Orthogonal Polynomial Expansion**

$$w_u(\mathbf{X}_u) = \sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}\\ j_1,\cdots,j_{|u|}\neq 0}} D_{u\mathbf{j}_{|u|}}\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u),\ \emptyset \neq u \subseteq \{1,\cdots,N\}$$

$$D_{u\mathbf{j}_{|u|}} := \int_{\mathbb{R}^N} \ln y(\mathbf{x})\psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u) f_{\mathbf{X}}(\mathbf{x})\,d\mathbf{x}$$

- **Exact**

$$y(\mathbf{X}) = \exp(w_\emptyset)\prod_{\emptyset\neq u\subseteq\{1,\cdots,N\}} \exp\left[\sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|}\\ j_1,\cdots,j_{|u|}\neq 0}} D_{u\mathbf{j}_{|u|}}\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)\right]$$

- **Approximate**

$$\bar{y}_{S,m}(\mathbf{X}) = \exp(w_\emptyset)\prod_{\substack{\emptyset\neq u\subseteq\{1,\cdots,N\}\\ 1\leq|u|\leq S}} \exp\left[\sum_{\substack{\mathbf{j}_{|u|}\in\mathbb{N}_0^{|u|},\left\|\mathbf{j}_{|u|}\right\|_\infty\leq m\\ j_1,\cdots,j_{|u|}\neq 0}} D_{u\mathbf{j}_{|u|}}\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)\right]$$

L-PDD

- **Univariate ($S=1$), mth-order L-PDD Approx.**

$$\bar{y}_{1,m}(\mathbf{X}) = \exp(w_{\emptyset}) \prod_{i=1}^{N} \exp\left[\sum_{j=1}^{m} D_{ij}\psi_{ij}(X_i)\right]$$

- **Statistical Moments**

$$\mathbb{E}\left[\bar{y}_{1,m}(\mathbf{X})\right] = \exp(w_{\emptyset}) \prod_{i=1}^{N} \mathbb{E}\left[\exp\left\{\sum_{j=1}^{m} D_{ij}\psi_{ij}(X_i)\right\}\right]$$

$$\begin{aligned}\mathbb{E}\left[\left(\bar{y}_{1,m}(\mathbf{X}) - \mathbb{E}\left[\bar{y}_{1,m}(\mathbf{X})\right]\right)^2\right] &= \exp(2w_{\emptyset}) \prod_{i=1}^{N} \mathbb{E}\left[\exp\left\{2\sum_{j=1}^{m} D_{ij}\psi_{ij}(X_i)\right\}\right] \\ &\quad - \exp(w_{\emptyset}) \prod_{i=1}^{N} \mathbb{E}\left[\exp\left\{\sum_{j=1}^{m} D_{ij}\psi_{ij}(X_i)\right\}\right]\end{aligned}$$

Coefficient Calculation

- **Dim.-Reduction Integration (Xu and Rahman, 2004)**

$$\hat{y}_R(\mathbf{x}) = \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \sum_{\substack{u\subseteq\{1,\cdots,N\}\\ |u|=R-k}} y(\mathbf{x}_u, \mathbf{c}_{-u})$$

$$C_{u\mathbf{j}_{|u|}} \cong \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \sum_{\substack{u\subseteq\{1,\cdots,N\}\\ |u|=R-k}} \times \int_{\mathbb{R}^{|u|}} y(\mathbf{x}_u, \mathbf{c}_{-u})\psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u) f_{\mathbf{X}_u}(\mathbf{x}_u)\, d\mathbf{x}_u$$

- Highly efficient when $R = S \ll N$ (also convergent)
- Polynomial complexity; 1D or 2D integration for univariate or bivariate approximation

Coefficient Calculation

- **Sampling (Monte Carlo Simulation)**

Input $\mathbf{x}^{(l)} \in \mathbb{R}^N$ $\underset{l=1,\cdots,L,\ L\in\mathbb{N}}{}$ $\rightarrow$ **COMPLEX SYSTEM** $\rightarrow$ Output $y(\mathbf{x}^{(l)})$

- F-PDD Coefficients

$$y_\emptyset \cong \frac{1}{L}\sum_{l=1}^{L} y(\mathbf{x}^{(l)}),$$

$$C_{u\mathbf{j}_{|u|}} \cong \frac{1}{L}\sum_{l=1}^{L} y(\mathbf{x}^{(l)})\psi_{u\mathbf{j}_{|u|}}(\mathbf{x}^{(l)}).$$

- L-PDD Coefficients

$$w_\emptyset \cong \frac{1}{L}\sum_{l=1}^{L} \ln y(\mathbf{x}^{(l)}),$$

$$D_{u\mathbf{j}_{|u|}} \cong \frac{1}{L}\sum_{l=1}^{L} \ln y(\mathbf{x}^{(l)})\psi_{u\mathbf{j}_{|u|}}(\mathbf{x}^{(l)}).$$

FGM (SiC-Al) Plate Modal Analysis

Random Input

- Particle Vol. Fraction (Beta RF)

$$
\begin{aligned}
\phi_p(\xi) &\cong F_p^{-1}\left[\Phi\left(\sum_{i=1}^{28} X_i\sqrt{\lambda_i}\psi_i(\xi)\right)\right] \\
\mu_p(\xi) &= 1-\xi/L \\
\sigma_p(\xi) &= (1-\xi/L)\xi/L \\
\Gamma_\alpha(\tau) &= \exp[-|\tau|/(0.125L)]
\end{aligned}
$$

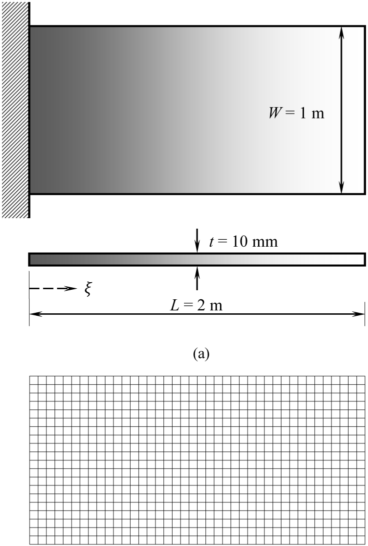
- Constituent Mat. Prop. (RVs)

$$
\begin{aligned}
E(\xi) &\cong E_p\phi_p(\xi)+E_m[1-\phi_p(\xi)] \\
\nu(\xi) &\cong \nu_p\phi_p(\xi)+\nu_m[1-\phi_p(\xi)] \\
\rho(\xi) &\cong \rho_p\phi_p(\xi)+\rho_m[1-\phi_p(\xi)]
\end{aligned}
$$

$E_p, E_m, \nu_p, \nu_m, \rho_p, \rho_m \rightarrow$ 6 LN variables

$$\mathbf{X} = \{X_1, \cdots, X_{34}\}^T \in \mathbb{R}^{34}$$

Coeff. calc. by dim.-red. integration

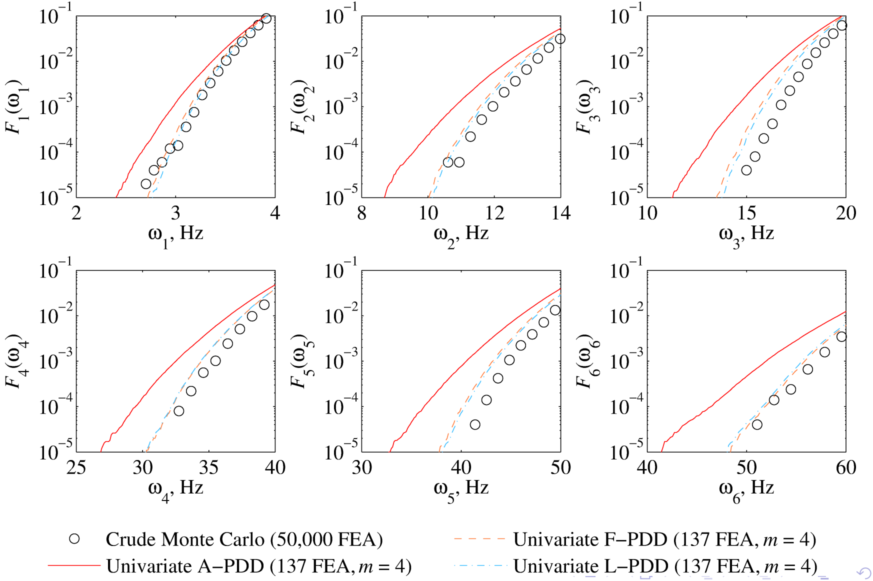


(a)

(b)

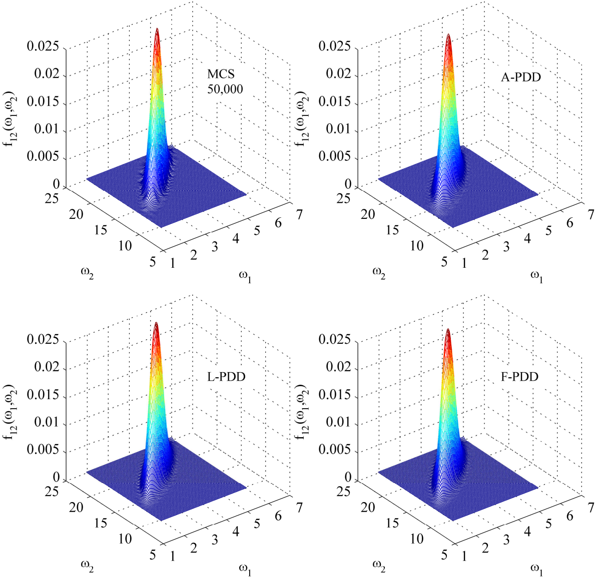
FGM Plate

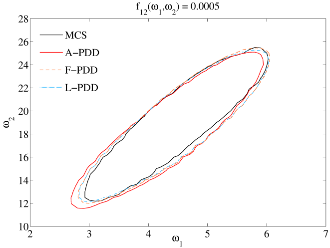
- **Marginal Distributions of Frequencies**



FGM Plate

- **Joint Densities of Frequencies**

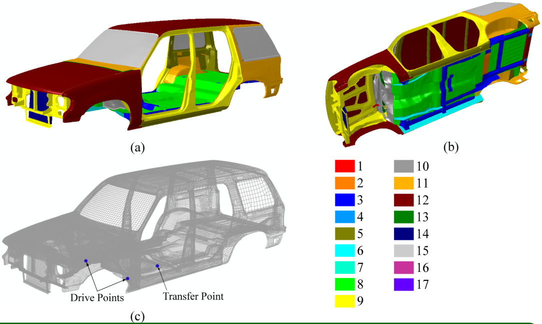




All PDDs: 137 FEA

MCS: 50,000 FEA

SUV Body in White (BIW)



Material	μ_E Gpa	μ_ρ kg/m^3	μ_s %
1	207	9500	1
2	207	9500	1
3	207	8100	1
4	207	29,260	1
5	207	29,260	1
6	207	37,120	1
7	207	9500	$-^{(a)}$
8	207	8100	$-^{(a)}$
9	207	8100	$-^{(a)}$
10	207	29,260	$-^{(a)}$
11	207	30,930	$-^{(a)}$
12	207	37,120	$-^{(a)}$
13	207	52,010	$-^{(a)}$
14	69	2700	$-^{(a)}$
15	69	2700	$-^{(a)}$
16	20	1189	$-^{(a)}$
17	200	1189	$-^{(a)}$

(a) The damping factors for materials 7-17 are equal to zero (deterministic).

500 Crude MCS for coeff. calc.

Random Input

- Young's modulus

$E_i(\mathbf{X}) = X_i$ GPa; $i = 1, \ldots, 17$

- Mass density

$\rho_i(\mathbf{X}) = X_{i+17}$ kg/m^3; $i = 1, \ldots, 17$

- Damping factor

$s_i(\mathbf{X}) = X_{i+34}$ %; $i = 1, \ldots, 6$

$\mathbf{X} = \{X_1, \ldots, X_N\}^T \in \mathbb{R}^{40}$

$X_i \sim$ Truncated Normal

$X_i \in [a_i, b_i]$

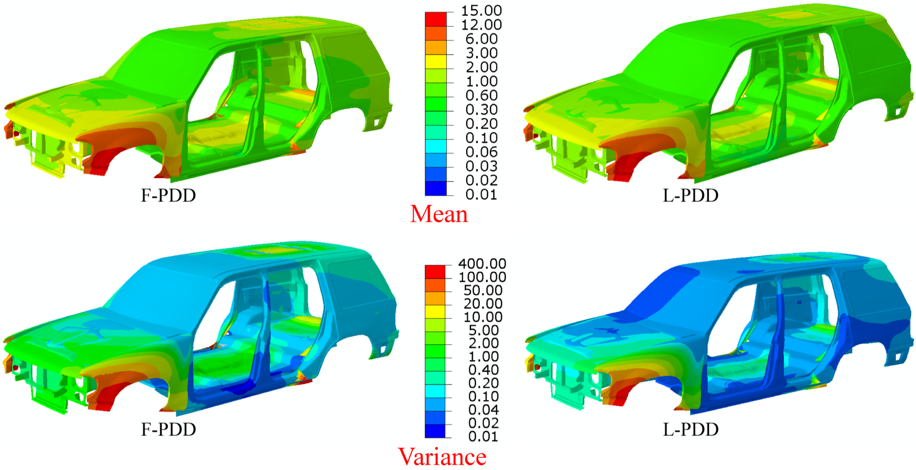
$a_i = 0.55\mu_i$; $b_i = 1.45\mu_i$

COV $v_i = 0.15$

$i = 1, \ldots, 40$

SUV BIW

- **Moments of 21st Mode Shape**



SUV BIW

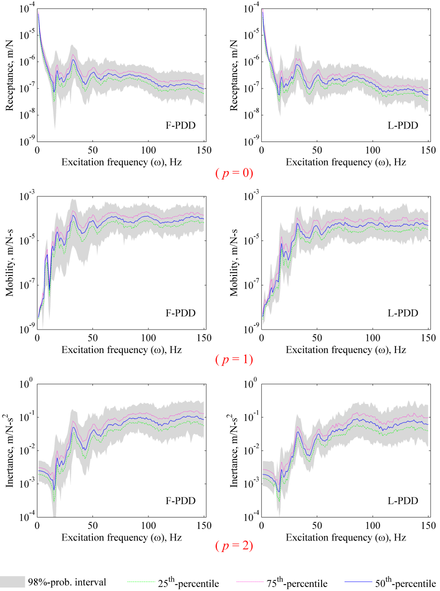
Frequency Response Functions

$$u_{p,d_1 d_2 t}(\omega) \simeq (i\omega)^p \sum_{k=1}^{K} \frac{[\phi_{k,d_1}+\phi_{k,d_2}]\phi_{k,t}}{[\Omega_k^2(1+is_k)-\omega^2]}$$

$i = \sqrt{-1}$,
K = retained eigenmodes,
$\phi_{k,d_1}, \phi_{k,d_2}$ = drive point vertical components of k^{th} eigenmode,
$\phi_{k,t}$ = transfer point vertical component of k^{th} eigenmode,
Ω_k = k^{th} eigenfrequency,
s_k = corresponding structural damping factor,
ω = excitation frequency.

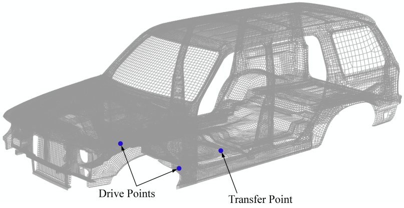
Global cutoff frequency = 300 Hz
$L = 10^4$

Real Parts



SUV BIW

- **Probabilities of Driver's Seat Acceleration**



$$A_{DS} = \left[\frac{1}{150} \sum_{j=1}^{150} \left\{ 1000 \alpha_j \, u_{2,d_1 d_2 t}(\omega_j) \right\}^2 \right]^{\frac{1}{2}}$$

	Interval of acceptable accelerations, m/s^{2} [(a)]					
Method	[0, 0.315]	[0.315, 0.63]	[0.5, 1]	[0.8, 1.6]	[1.25, 2.5]	[2, ∞)
A-PDD	7.4×10^{-1}	2.6×10^{-1}	3.1×10^{-3}	0	0	0
F-PDD	6.9×10^{-1}	2.7×10^{-1}	3.6×10^{-2}	3.2×10^{-3}	7.3×10^{-4}	4.1×10^{-4}
L-PDD	8.4×10^{-1}	1.1×10^{-1}	3.4×10^{-2}	8.8×10^{-3}	2.1×10^{-3}	4.3×10^{-4}

(a) From International Standard ISO 2631.

Conclusions

- Two mult. decomp., F-PDD and L-PDD, developed; exploit hidden multiplicative structure, if it exists
- New relationship between ANOVA and F-PDD component functions developed
- Multiplicative PDD methods can be more accurate than additive PDD, at the same cost
- The new methods were applied in solving large-scale practical engineering problems

Future Work

- Adaptive and sparse algorithms in conjunction with the PDD methods
- Hybrid PDD method to account for the combined effects of the multiplicative and additive structures