# Adaptive Polynomial Dimensional Decompositions for Uncertainty Quantification in High Dimensions

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# Outline

# Polynomial Dim. Decomposition (Rahman, 2008)

$$\text{Input } \mathbf{X} \in \mathbb{R}^N \rightarrow \boxed{\textbf{COMPLEX SYSTEM}} \rightarrow \text{Output } y(\mathbf{X}) \in \mathbb{R}$$

$$\mathbf{X} \sim (\Omega, \mathcal{F}, P); \; y \in \mathcal{L}_2(\Omega, \mathcal{F}, P)$$

- **ANOVA Dimensional Decomposition**

$$y(\mathbf{X}) = \sum_{u \subseteq \{1,\cdots,N\}} y_u(\mathbf{X}_u)$$

- **PDD**

$$\boxed{y(\mathbf{X}) \;:=\; y_\emptyset + \sum_{\emptyset \neq u \subseteq \{1,\cdots,N\}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|} \\ j_1,\cdots,j_{|u|} \neq 0}} C_{u\mathbf{j}_{|u|}} \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)}$$

$\psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u) \rightarrow$ product of univariate orthonormal polynomials

$$y_\emptyset := \int_{\mathbb{R}^N} y(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}; \quad C_{u\mathbf{j}_{|u|}} := \int_{\mathbb{R}^N} y(\mathbf{x}) \psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u) f_{\mathbf{X}}(\mathbf{x})\, d\mathbf{x}$$

# PDD Approximation

- **$S$-variate, $m$th-order PDD Approximation**

$$\tilde{y}_{S,m}(\mathbf{X}) = y_\emptyset + \sum_{\substack{\emptyset \neq u \subseteq \{1,\cdots,N\} \\ 1 \leq |u| \leq S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \|\mathbf{j}_{|u|}\|_\infty \leq m \\ j_1,\cdots,j_{|u|} \neq 0}} C_{u\mathbf{j}_{|u|}} \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)$$

## An Example ($S = 2$)

$$\tilde{y}_{2,m}(\mathbf{X}) = y_\emptyset + \sum_{i=1}^{N} \sum_{j=1}^{m} C_{ij} \psi_{ij}(X_i) + \sum_{i_1 < i_2}^{N} \sum_{j_2=1}^{m} \sum_{j_1=1}^{m} C_{i_1 i_2 j_1 j_2} \psi_{i_1 j_1}(X_{i_1}) \psi_{i_2 j_2}(X_{i_2})$$

- How to select truncation parameters: $S$, $m$?
- Given $S$, all component functions may not be needed
- $m$ can be different for different component functions

# Global Sensitivity Analysis

- **Sensitivity Indices for** $\emptyset \neq u \subseteq \{1, \cdots, N\}$ **and** $m_u$

$$\boxed{\tilde{G}_{u,m_u} := \frac{\displaystyle\sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \|\mathbf{j}_{|u|}\|_\infty \leq m_u \\ j_1, \cdots, j_{|u|} \neq 0}} C^2_{u\mathbf{j}_{|u|}}}{\displaystyle\sum_{\substack{\emptyset \neq u \subseteq \{1, \cdots, N\} \\ 1 \leq |u| \leq S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \|\mathbf{j}_{|u|}\|_\infty \leq m_u \\ j_1, \cdots, j_{|u|} \neq 0}} C^2_{u\mathbf{j}_{|u|}}}}$$

$$\Delta \tilde{G}_{u,m_u} = \frac{\tilde{G}_{u,m_u} - \tilde{G}_{u,m_u-1}}{\tilde{G}_{u,m_u-1}}$$

- $\tilde{G}_{u,m_u}$: $m_u$th-order contribution of $y_u(\mathbf{X}_u)$ to $\sigma^2$
- $\Delta \tilde{G}_{u,m_u}$: change in $\tilde{G}_{u,m_u}$ for an increase in $m_u$

# Adaptive-Sparse PDD

- **Fully Adaptive-Sparse PDD**

$$\bar{y}(\mathbf{X}) := y_\emptyset + \sum_{\emptyset \neq u \subseteq \{1,\cdots,N\}} \sum_{m_u=1}^{\infty} \sum_{\substack{\|\mathbf{j}_{|u|}\|_\infty = m_u,\, j_1,\cdots,j_{|u|} \neq 0 \\ \tilde{G}_{u,m_u} > \epsilon_1,\, \Delta\tilde{G}_{u,m_u} > \epsilon_2}} C_{u\mathbf{j}_{|u|}} \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)$$

$$\bar{\sigma}^2 = \sum_{\emptyset \neq u \subseteq \{1,\cdots,N\}} \sum_{m_u=1}^{\infty} \sum_{\substack{\|\mathbf{j}_{|u|}\|_\infty = m_u,\, j_1,\cdots,j_{|u|} \neq 0 \\ \tilde{G}_{u,m_u} > \epsilon_1,\, \Delta\tilde{G}_{u,m_u} > \epsilon_2}} C^2_{u\mathbf{j}_{|u|}}$$

- $\tilde{G}_{u,m_u} > \epsilon_1$: degree of interaction
- $\Delta\tilde{G}_{u,m_u} > \epsilon_2$: order of polynomial
- If $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$, then $\bar{y}(\mathbf{X}) \to y(\mathbf{X})$

# Adaptive-Sparse PDD

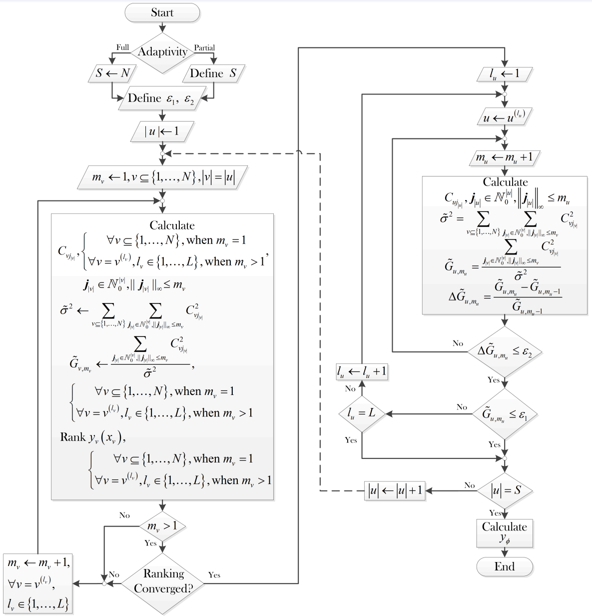
- **Partially Adaptive-Sparse PDD**

$$\bar{y}_S(\mathbf{X}) := y_\emptyset + \sum_{\substack{\emptyset \neq u \subseteq \{1,\cdots,N\} \\ 1 \leq |u| \leq S}} \sum_{m_u=1}^{\infty} \sum_{\substack{\|\mathbf{j}_{|u|}\|_\infty = m_u,\, j_1,\cdots,j_{|u|} \neq 0 \\ \tilde{G}_{u,m_u} > \epsilon_1,\, \Delta\tilde{G}_{u,m_u} > \epsilon_2}} C_{u\mathbf{j}_{|u|}} \psi_{u\mathbf{j}_{|u|}}(\mathbf{X}_u)$$

$$\bar{\sigma}_S^2 = \sum_{\substack{\emptyset \neq u \subseteq \{1,\cdots,N\} \\ 1 \leq |u| \leq S}} \sum_{m_u=1}^{\infty} \sum_{\substack{\|\mathbf{j}_{|u|}\|_\infty = m_u,\, j_1,\cdots,j_{|u|} \neq 0 \\ \tilde{G}_{u,m_u} > \epsilon_1,\, \Delta\tilde{G}_{u,m_u} > \epsilon_2}} C_{u\mathbf{j}_{|u|}}^2$$

- $\tilde{G}_{u,m_u} > \epsilon_1$: degree of interaction
- $\Delta\tilde{G}_{u,m_u} > \epsilon_2$: order of polynomial
- If $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$, then $\bar{y}_S(\mathbf{X}) \to \tilde{y}_{S,m}(\mathbf{X})$ as $m \to \infty$

# Numerical Implementation




- Ranking convergence
  - ranking discrepancy ratio

$$\frac{N_{m_u}}{L} < \epsilon_3$$

- Full ranking
  - $m_u \leftarrow m_u + 1$
  - calculate $C_{u\mathbf{j}_{|u|}}$
  - for each $y_u(\mathbf{X}_u)$, $u \subseteq \{1, \cdots, N\}$
- Reduced ranking
  - $m_u \leftarrow m_u + 1$
  - calculate $C_{u\mathbf{j}_{|u|}}$
  - for each $y_u(\mathbf{X}_u)$, with $\tilde{G}_{u,m_u} > \epsilon_1$

**Reduced ranking can be highly efficient**

# Computational Efforts

- **Total number of coefficients**

$$\tilde{K}_{S,m} = 1 + \sum_{\substack{\emptyset \neq u \subseteq \{1,\cdots,N\} \\ 1 \leq |u| \leq S}} \sum_{\substack{\mathbf{j}_{|u|} \in \mathbb{N}_0^{|u|}, \|\mathbf{j}_{|u|}\|_\infty \leq m \\ j_1,\cdots,j_{|u|} \neq 0}} 1 = \sum_{k=0}^{S} \binom{N}{k} m^k$$

$$\bar{K} = 1 + \sum_{\emptyset \neq u \subseteq \{1,\cdots,N\}} \sum_{m_u=1}^{\infty} \sum_{\tilde{G}_{u,m_u} > \epsilon_1, \Delta \tilde{G}_{u,m_u} > \epsilon_2} \left[ m_u^{|u|} - (m_u - 1)^{|u|} \right]$$

$$\bar{K}_S = 1 + \sum_{\substack{\emptyset \neq u \subseteq \{1,\cdots,N\} \\ 1 \leq |u| \leq S}} \sum_{m_u=1}^{\infty} \sum_{\tilde{G}_{u,m_u} > \epsilon_1, \Delta \tilde{G}_{u,m_u} > \epsilon_2} \left[ m_u^{|u|} - (m_u - 1)^{|u|} \right]$$

- If $\epsilon_1 \to 0$, and $\epsilon_2 \to 0$, then $\bar{K}_S \to \tilde{K}_{S,m}$ as $m \to \infty$
- If $m$ is the largest polynomial order in adaptive-sparse PDD, then $\bar{K}_S \leq \tilde{K}_{S,m}$, for $\epsilon_1 > 0$, and $\epsilon_2 > 0$

# Coefficient Calculation

- **Dim.-Reduction Integration (Xu and Rahman, 2004) with Full-grid and Sparse-grid**

$$C_{u\mathbf{j}_{|u|}} \cong \sum_{k=0}^{R}(-1)^k \binom{N-R+k-1}{k} \sum_{\substack{u\subseteq\{1,\cdots,N\}\\|u|=R-k}} \times \int_{\mathbb{R}^{|u|}} y(\mathbf{x}_u,\mathbf{c}_{-u})\psi_{u\mathbf{j}_{|u|}}(\mathbf{x}_u) f_{\mathbf{X}_u}(\mathbf{x}_u)\,d\mathbf{x}_u$$

- **Quasi Monte Carlo Simulation**

1. Generate a low-discrepancy point set $\mathcal{P}_L := \left\{\mathbf{u}^{(k)} \in [0,1]^N,\ k=1,\cdots,L\right\};\ L\in\mathbb{N}$
2. Map $\mathbf{u}^{(k)}$ to the input sample $\mathbf{x}^{(k)} \in \mathbb{R}^N$
3. Approximate

$$C_{u\mathbf{j}_{|u|}} \cong \frac{1}{L}\sum_{k=1}^{L} y\left(\mathbf{x}^{(k)}\right)\psi_{u\mathbf{j}_{|u|}}\left(\mathbf{x}_u^{(k)}\right)$$

# FGM (SiC-Al) Plate Modal Analysis ($N = 34$)

## Random Input

- Particle Vol. Fraction (Beta RF)

$$
\begin{aligned}
\phi_p(x) &\cong F_p^{-1}\left[\Phi\left(\sum_{i=1}^{28} X_i\sqrt{\lambda_i}\psi_i(x)\right)\right] \\
\mu_p(x) &= 1 - x/L \\
\sigma_p(x) &= (1 - x/L)x/L \\
\Gamma_\alpha(\tau) &= \exp[-|\tau|/(0.125L)]
\end{aligned}
$$

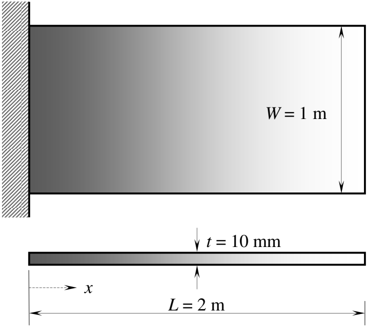
- Constituent Mat. Prop. (RVs)

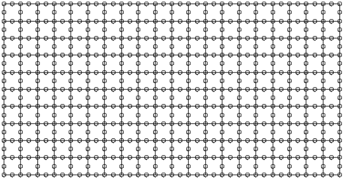
$$
\begin{aligned}
E(x) &\cong E_p\phi_p(x) + E_m[1 - \phi_p(x)] \\
\nu(x) &\cong \nu_p\phi_p(x) + \nu_m[1 - \phi_p(x)] \\
\rho(x) &\cong \rho_p\phi_p(x) + \rho_m[1 - \phi_p(x)]
\end{aligned}
$$

$E_p, E_m, \nu_p, \nu_m, \rho_p, \rho_m \to 6$ LN variables

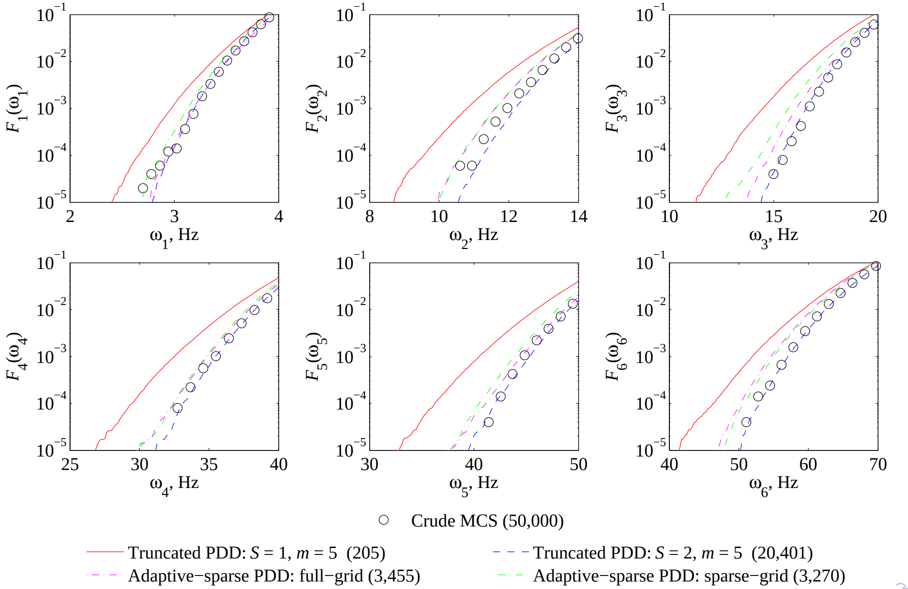
$$\boldsymbol{X} = \{X_1, \cdots, X_{34}\}^T \in \mathbb{R}^{34}$$

Coeff. calc. by dim.-red. integration

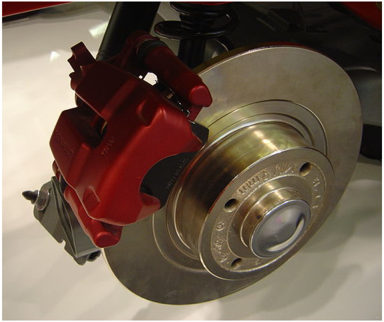


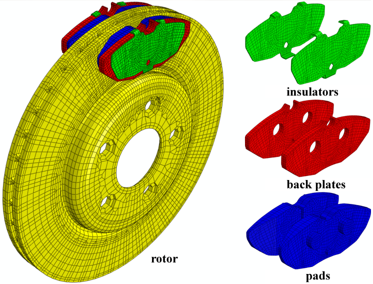



# FGM: Marginal Distributions of Frequencies



# Disk Brake System: Brake Squeal Analysis



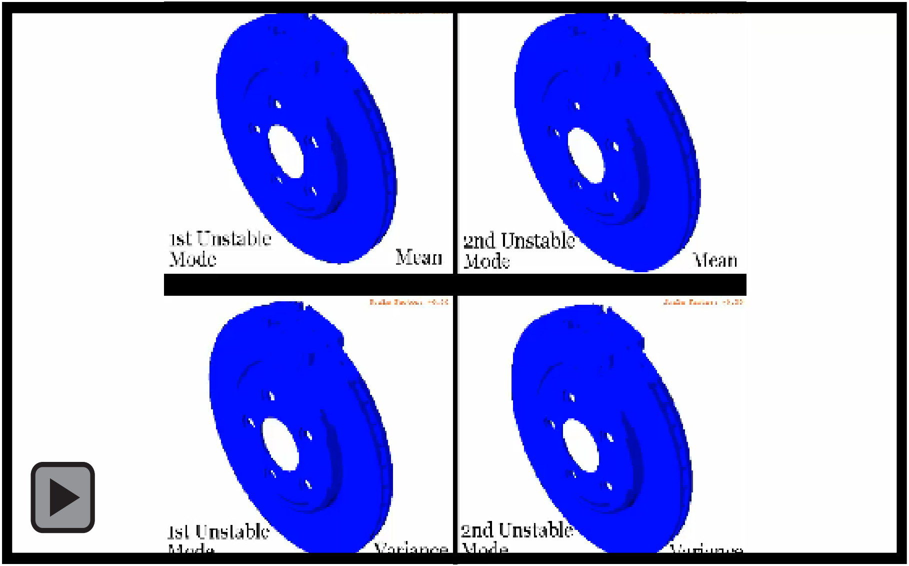



| RV | Quantity | $a_i$ | $b_i$ |
|---|---|---|---|
| $X_1$ | $\rho_{\text{rotor}}$, kg/m$^3$ | 5329 | 9071 |
| $X_2$ | $\rho_{\text{back plate}}$, kg/m$^3$ | 5788 | 9851 |
| $X_3$ | $\rho_{\text{insulator}}$, kg/m$^3$ | 5788 | 9851 |
| $X_4$ | $\rho_{\text{pad}}$, kg/m$^3$ | 1858 | 3162 |
| $X_5$ | $E_{\text{rotor}}$, GPa | 92.52 | 157.5 |
| $X_6$ | $E_{\text{back plate}}$, GPa | 153.2 | 260.8 |
| $X_7$ | $E_{\text{insulator}}$, GPa | 153.2 | 260.8 |
| $X_8$ | $E_{1,\text{pad}}$, GPa | 4.068 | 6.924 |
| $X_9$ | $E_{2,\text{pad}}$, GPa | 4.068 | 6.924 |
| $X_{10}$ | $E_{3,\text{pad}}$, GPa | 1.468 | 2.498 |
| $X_{11}$ | $G_{12,\text{pad}}$, GPa | 1.917 | 3.263 |
| $X_{12}$ | $G_{13,\text{pad}}$, GPa | 0.873 | 1.486 |
| $X_{13}$ | $G_{23,\text{pad}}$, GPa | 0.873 | 1.486 |
| $X_{14}$ | $p$, kg/mm$^2$ | 370.1 | 629.9 |
| $X_{15}$ | $\omega$, rad/s | 3.701 | 6.299 |
| $X_{16}$ | $\mu$ | 0.50 | 0.70 |

500 Quasi MCS for coeff. calc.

# Disk Brake System

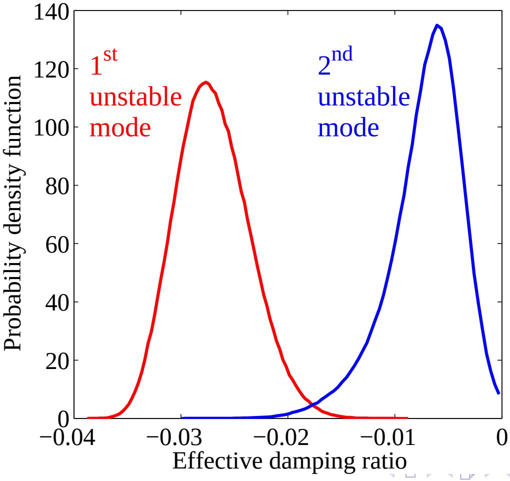
- **Moments of unstable mode shapes**



# Disk Brake System

- **PDF of effective damping ratios**

$$R_i = -2\mathrm{Re}\left[\lambda_u^{(i)}\left(\boldsymbol{X}\right)\right]/\mathrm{Im}|\lambda_u^{(i)}\left(\boldsymbol{X}\right)|$$



## Conclusions

- An adaptive-sparse algorithm for PDD developed
- Global sensitivity analysis employed to retain component functions
- Does not require PDD truncation parameters to be assigned *a priori* or arbitrarily
- Achieves a desired level of accuracy with fewer coefficients
- High-dimensional complex engineering problems solved using the new method